SOLUTIONS TO QUIZ - MATH 170 - SUMMER SESSION I (2012)

Name:

INSTRUCTIONS:

1. The duration of this quiz is 1 hour - from 4:05 p.m. till 5:05 p.m.
2. No calculators, electronic watches, cellphones etc. will be allowed in this quiz.
3. This is a "closed book" quiz, meaning that, looking at notes or computers or taking the help of cheat sheets is not permitted.
4. As per Penn's Academic Integrity, any form of cheating is strongly discouraged and shall not be tolerated.
5. There are seven pages in this quiz. Make sure your copy has all of them.
6. There are six questions in this quiz. All the questions are compulsory.
7. Show all your work, and right down your reasoning as clearly as possible. State any known theorems or results that you use while working out a problem.
8. If you have any question about any of the problems, please ask your instructor and not any other student.

(1) Write the definitions of the following terms:

(a) Twin primes

Twin primes are pairs of natural numbers $(p, p+2)$ such that both p and $p+2$ are primes.

(b) Rational number

A rational number is a number that can be expressed as a fraction $\frac{p}{q}$ where both p and q are integers and $q \neq 0$.

(c) Fibonacci sequence

It is a sequence of numbers a_n that satisfies the condition $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1, a_2 = 1$. Therefore, the first couple of terms of the sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34$ etc.

Date: June 7, 2012.

(d) Intersection of two sets

For two sets A and B, their intersection is defined as the set whose elements are those that are elements of *both* A and B. $A \cap B = \{x : x \text{ is in } A \text{ and } x \text{ is in } B\}$.

(2) Write the statement of the Prime Factorization Theorem.

The Prime Factorization Theorem states that every natural number $n > 1$ can be expressed as a product of prime numbers, with the number of prime factors and the prime factors themselves being unique up to the order in which they appear in the product.

Based on that, write the following numbers completely as a product of their prime factors.

(a) $126 = 2 \times 3^2 \times 7$.

(b) $3125 = 5^5$.

(c) $352 = 2^5 \times 11$.

(3) Do any ONE of the following two parts (do not do both):

(a) If a, b and c are three integers such that b is divisible by a and c is divisible by b, then show that c is divisible by a.

If b is divisible by a, then there is an integer k for which $b = k \cdot a$. If c is divisible by b, then there is an integer m for which $c = m \cdot b$.

Thus, $c = m \cdot b = m \cdot k \cdot a$. So, c is divisible by a.

(b) Show that $24^{55} \equiv 5^2 \times 2^2 \mod 19$. State clearly if you use any theorem.

Since 19 is prime and doesn't divide 24, by Fermat's little theorem, $24^{19-1} \equiv 1 \mod 19$. So, $24^{18} \equiv 1 \mod 19$. Taking cubes, $24^{54} \equiv 1 \mod 19$. So, $24^{55} \equiv 24 \mod 19$. But, 24 gives a remainder of 5 when divided by 19. Then, $24^{55} \mod 19 = 5$.

Next, $5^2 \times 2^2 \mod 19 = 100 \mod 19 = (95 + 5) \mod 19 = (19 \times 5 + 5) \mod 19 = 5$. Therefore, $24^{55} \equiv 5^2 \times 2^2 \mod 19$.

(4) Suppose someone asked you to toss a coin three times, and then asked you what the probability of getting at least one tail is.

(a) Write down the universal set in this case.

$\Omega = \{(H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, H, T), (T, T, H), (T, T, T)\}.$

(b) Describe the event in English and use set notation to write down all its elements.

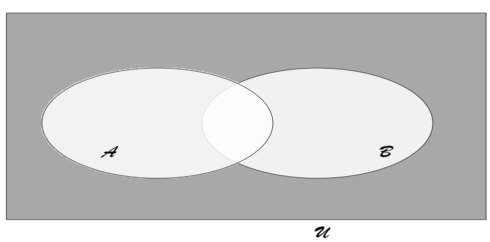
$E =$ the event of getting at least one tail
$= \{(H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, H, T), (T, T, H), (T, T, T)\}.$

(c) Now, find out the probability of occurrence of that event.

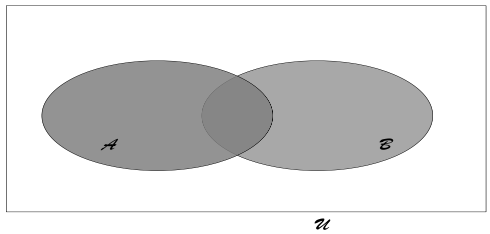
$P(E) = \frac{n(E)}{n(\Omega)} = \frac{7}{8}.$

(5) Let A and B be two sets contained in a universal set $\mathcal{U}$. Draw Venn diagrams and shade regions that represent:

(a) $(A - B)^c - (A \cup B)$



(b) $(A \cap B) \cup (A \triangle B)$



(6) Do any ONE of the following two parts (do not do both):

(a) Show, by method of contradiction, that $\sqrt{3}$ is an irrational number. Using this fact (and perhaps, also method of contradiction), show that $\sqrt{3} + \frac{1}{\sqrt{3}}$ is also irrational.

For the first part, suppose $\sqrt{3}$ is rational. Then, we may write, $\sqrt{3}$ as a fraction $\frac{p}{q}$, where p and q are integers having *no common factors* and $q \neq 0$. So, squaring both sides,

$$3 = \frac{p^2}{q^2},$$
$$3q^2 = p^2,$$

which means 3 divides p^2. Since 3 is a prime, this implies that 3 divides p. So we may write $p = 3 \cdot k$. Plugging this back in to the above equation, we get,

$$3q^2 = (3k)^2 = 9k^2,$$
$$q^2 = 3k^2.$$

Therefore, 3 divides q^2. As before, this means that 3 divides q. Therefore 3 divides both p and q which contradicts the fact that p and q have no common factors. So, $\sqrt{3}$ is irrational.

Next, assume that $\sqrt{3} + \frac{1}{\sqrt{3}}$ is a rational number. Then we can write it as a fraction $\frac{m}{n}$, where m and n are integers with no common factors and $n \neq 0$. So,

$$\sqrt{3} + \frac{1}{\sqrt{3}} = \frac{m}{n}.$$

Multiplying throughout by $\sqrt{3}$ we get,

$$\begin{aligned} \sqrt{3} \cdot \sqrt{3} + \sqrt{3} \cdot \frac{1}{\sqrt{3}} &= \frac{\sqrt{3}m}{n}, \\ \implies 3 + 1 &= \frac{\sqrt{3}m}{n}, \\ \implies \sqrt{3} &= \frac{4n}{m}, \end{aligned}$$

which is a contradiction as $\sqrt{3}$ is irrational. Hence, $\sqrt{3} + \frac{1}{\sqrt{3}}$ is also irrational.

(b) Show, using induction, that $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Step 1: We wish to check that the statement is true for $n = 1$. Here, the left hand side is $1^2 = 1$ and the right hand side is $\frac{1(1+1)(2(1)+1)}{6} = \frac{1 \times 2 \times 3}{6} = 1$. Hence, the statement is indeed true for $n = 1$.
Step 2: We assume that the statement is true for $n = m$. That is, we assume,

$$1^2 + 2^2 + 3^2 + ... + m^2 = \frac{m(m+1)(2m+1)}{6}.$$

Step 3: We need to prove the statement for $n = m + 1$. That is, we need to prove that,

$$\begin{aligned} 1^2 + 2^2 + 3^2 + ... + (m+1)^2 &= \frac{(m+1)(m+1+1)(2(m+1)+1)}{6} \\ &= \frac{(m+1)(m+3)(2m+3)}{6}. \end{aligned}$$

Now,

$$\begin{aligned}
1^2+2^2+3^2+...+(m+1)^2 &= 1^2+2^2+3^2+...+m^2+(m+1)^2 \\
&= \frac{m(m+1)(2m+1)}{6}+(m+1)^2 \\
&= (m+1)\left[\frac{m(2m+1)}{6}+(m+1)\right] \\
&= (m+1)\left[\frac{2m^2+m}{6}+\frac{6m+6}{6}\right] \\
&= (m+1)\left[\frac{2m^2+m+6m+6}{6}\right] \\
&= (m+1)\left[\frac{2m^2+7m+6}{6}\right] \\
&= (m+1)\left[\frac{(m+2)(2m+3)}{6}\right] \\
&= \frac{(m+1)(m+2)(2m+3)}{6},
\end{aligned}$$

which proves our statement for step 3. Hence, by induction, $1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$, for all natural numbers n.