SOLUTIONS TO HOMEWORK 1 - MATH 170, SUMMER SESSION I (2012)

(1) In class, we had used Pigeonhole Principle to show that if we choose any 6 numbers from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, at least two of those numbers must add up to 10. Along the same lines,

(a) can you show the more general statement that for any n, if we choose any $n+1$ numbers from $\{1, 2, 3, ..., 2n-1\}$, at least two of those chosen numbers add up to $2n$?

Solution: We solve this problem using Pigeonhole Principle as follows. We think of the $n+1$ chosen numbers as our "envelopes". The "pigeonholes" are collections of numbers that add up to $2n$. Thus, the pigeonholes are

$$\{1, 2n-1\}, \{2, 2n-2\}, \{3, 2n-3\}, \cdots, \{n-1, n+1\}, \{n\}.$$

Clearly there are n pigeonholes and $n+1$ envelopes. Thus, if we were to distribute the envelopes in the pigeonholes according as sending each number to the collection it belongs to, then by pigeonhole principle, at least one of the collections must have more than one envelope. Therefore, there must be at least two numbers among the chosen $n+1$ that add up to $2n$.

(b) What if we were to choose n numbers from $\{1, 2, 3, ..., 2n-1\}$? Does the same conclusion hold? That is, will there still be at least two numbers from the chosen ones that add up to $2n$? If you think yes, show with good arguments. If you think no, give an example where it does not hold.

Solution: No, the same conclusion does not hold. As an example, we could have chosen the n numbers $\{1, 2, ..., n\}$. Clearly no two of them will add up to $2n$. To see this clearly, note that the largest two numbers among these are $n-1$ and n and these add up to $2n-1$. Since these are the largest two numbers, all other number pairs must add up to less than $2n-1$.

(2) (Exercise 14, Page 60) You have 10 pairs of socks, 5 black pairs and 5 blue pairs, but they are not paired up. Instead, they are all mixed up in a drawer. It's early in the morning and you don't want to turn on the lights in your dark room. How many socks must you pull out to guarantee that you have a pair of one color? How

many must you pull out to have two good pairs (each pair is the same color)? How many must you pull out to be certain that you have a pair of black socks?

Solution: For the first question, we can easily guess that we must pull out 3 socks to guarantee a good pair. We can see this using Pigeonhole as follows. We call the colors of the three socks our "envelopes", and we form two "pigeonholes", BLACK and BLUE. We send each of our envelopes into the pigeonhole of its corresponding color. Since there are more envelopes than pigeonholes, at least one of the pigeonholes must hold more than one sock color. Thus, at least two of the chosen socks are of the same color.

For the second question, the answer is five. Again, we call the colors of the five socks our "envelopes", and we form two "pigeonholes", BLACK and BLUE. We send each of our envelopes into the pigeonhole of its corresponding color. Since the number of envelopes is more than twice the number of pigeonholes, at least one pigeonhole has more than two envelopes. So, at least one pigeonhole has 3, 4 or 5 envelopes. In case it has 4 or 5 envelopes, clearly we have to pairs of socks of the same color. If it has three envelopes, the other pigeonhole has 2, and so you again have two good pairs of socks.

For the third question, clearly any number less than or equal to 10 cannot work, since there are 10 blue socks and there is a chance that you had pulled out all ten of them at one go. Again, 11 cannot work either, since, in the worst case scenario, you can pull out 10 blue and 1 black, leaving you 1 short of making a black pair. The correct answer here is 12. Since there are only 10 blue socks, choosing 12 of them will ensure that you have at most 10 blue socks, leaving you with at least 2 black socks.

(3) This is the question that Leonardo of Pisa (or Fibonacci) asked himself, and this came up with the famous Fibonacci numbers.

Suppose we have a pair of baby rabbits: one male and one female. Let us assume that the rabbits cannot reproduce until they are one month old, and that they have a one-month gestation period. Once they start reproducing, they produce a pair of bunnies each month (one of each sex). Let us assume here that no pair ever dies.

(a) Fill in (out?) the following table:

Time in Months	Number of Pairs
0 (Start)	1
1	1
2	
3	
4	
5	
6	
7	
8	

Solution:

Time in Months	Number of Pairs
0 (Start)	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34

(b) Guess a pattern that the quantity "the number of pairs of bunnies at the end of a month" follows. Explain clearly and in words why you predict this pattern (your explanation should be based on the information you have about how quickly the baby bunnies grow and what the gestation period is... not on the table above).

Solution: From the table above, we are inclined to guess that the pattern is the Fibonacci Sequence. Here is the reason why this is true. At the end of the nth month, the total number of bunny pairs is equal to the sum of the number of adult rabbit pairs and the number of baby rabbit pairs. Now, the number of adult rabbit pairs is exactly equal to the total number of bunny pairs at the end of the $(n-1)$th month (since the ones that were babies in the previous month grew up into adults by the end of the current month). Also, the number of baby rabbit pairs at the end of the nth month is exactly equal to the number of adult rabbit pairs at the end of the $(n-1)$th month (since each adult rabbit pair produced a baby rabbit pair at the end of the nth month). But, by the same reasoning as before, the number of adult rabbit pairs at the end of the $(n-1)$th month is exactly equal to the total number of bunny pairs at the end of the $(n-2)$th month. Thus, the total number of bunny pairs at the end of the nth month is equal to the sum of the total number of bunny pairs at the end of the $(n-1)$th month and total number of bunny pairs at the end of the $(n-2)$th month, which explains its Fibonacci pattern.

(4) (Exercise 16, Page 73) Express each of the following natural numbers as a sum of distinct, nonconsecutive Fibonacci numbers: 52, 143, 13, 88.

Solution:

$$\begin{aligned}
52 &= 34 + 18 \\
&= \underline{34 + 13 + 5}. \\
143 &= 89 + 54 \\
&= 89 + 34 + 20 \\
&= 89 + 34 + 13 + 7 \\
&= \underline{89 + 34 + 13 + 5 + 2}. \\
13 &= 8 + 5 \\
&= \underline{8 + 3 + 2}. \\
88 &= 55 + 33 \\
&= 55 + 21 + 12 \\
&= 55 + 21 + 8 + 4 \\
&= \underline{55 + 21 + 8 + 2 + 1 + 1}.
\end{aligned}$$

(5) We showed in class that the square of any natural number is either divisible by 4 or leaves a remainder of 1 when divided by 4. Show now that the square of any natural number is either divisible by 3 or leaves a remainder of 1 when divided by 3.

Solution: For any natural number, n, we can use division algorithm of n with 3 to write n as

$$n = 3q + r,$$

where q is the quotient when dividing n by 3, and r is the remainder, with $0 \leq r \leq 3 - 1$. i.e. $0 \leq r \leq 2$. So, r can take only 3 values, 0, 1 and 2.

Case 1: $r = 0$
In this case, $n = 3q$, and so $n^2 = 9q^2 = 3(3q^2)$, which is divisible by 3.

Case 2: $r = 1$
In this case, $n = 3q + 1$, and so $n^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$. Thus, when divided by 3, n^2 leaves a remainder of 1.

Case 3: $r = 2$
In this case, $n = 3q+2$, and so $n^2 = (3q+2)^2 = 9q^2+12q+4 = 9q^2+12q+3+1 = 3(3q^2+4q+1)+1$. Thus, when divided by 3, n^2 leaves a remainder of 1.

Hence, in all the three cases, n^2 is either divisible by 3 or leaves a remainder of 1 when divided by 3.

(6) True or False: If a and b are natural numbers with a divides b and b divides a, then $a = b$.

If you think this statement is true, show this mathematically. If you think it is false, then give an example where this does not hold.

Solution: Since a divides b, there is a natural number k such that $b = k \cdot a$. Since b divides a, there is a natural number m such that $a = m \cdot b$. So,

$$a = m \cdot b = m \cdot k \cdot a.$$

Canceling a from both the sides, we get $mk = 1$, which is possible only if m and k are both 1 (since m and k are natural numbers). So, $a = b$.

(7) (a) Write down the first 8 twin prime pairs. Also, next to each pair, write down its sum. (For example, the first twin prime pair is (3,5) and their sum is 3+5=8, and so on).

Solution:

Twin Prime Pairs	Sum
(3,5)	8
(5,7)	12
(11,13)	24
(17,19)	36
(29,31)	60
(41,43)	84
(59,61)	120
(71,73)	144
(101,103)	204

(b) We showed in class that if p and $p+2$ are twin primes, with $p > 3$, then 3 divides $p+1$. Using this fact and other knowledge you have about primes and integers in general, show also that if p and $p+2$ are twin primes, with $p > 3$, then their sum is divisible by 12.

Solution: If $p, p+2$ is a twin prime pair with $p > 3$, then as we proved in class, $p+1$ is divisible by 3. Also, if p is any prime number bigger than 2, p will be an odd number. This means that $p+1$ is even. Thus, $p+1$ is divisible by both 3 and 2, which means it is divisible by 6.

Now, $p + p + 2 = 2(p+1)$. So, as $p+1$ is divisible by 6, and we multiply it by 2 to get the sum $p+p+2$, the sum must be divisible by 12.

(8) Compute the following:

(a) 5^3 + (148-71)×43 $\mod 16$

Solution: $5^3 = 125 = 112 + 13 = 16 \times 7 + 13$. Therefore, $5^3 \mod 16 = 13$. Next, $148 - 71 = 77 = 64 + 13 = 16 \times 4 + 13$, and so, $(148 - 71) \mod 16 = 13$. $43 = 32 + 11 = 16 \times 2 + 11$, and so, $43 \mod 16 = 11$.

So, $5^3 + (148 - 71) \times 43 \mod 16 = 13 + 13 \times 11 \mod 16 = 13 + 143 \mod 16 = 156 \mod 16 = (144 + 12) \mod 16 = (16 \times 9 + 12) \mod 16 = 12$.

(b) $6^{102} \mod 11$

Solution: As we had discussed in class, $6^2 = 36 = 33 + 3$, and so, $6^2 \mod 11 = 3$. Thus, $6^{102} \mod 11 = 3^{51} \mod 11$. We now look for powers of 3 that are equal to 1 modulo 11.

Now, note that, $3^2 = 9$,
$3^3 = 27 = 22 + 5$ and so, $3^3 \mod 11 = 5$,
$3^4 = 81 = 77 + 4$ and so, $3^4 \mod 11 = 4$,
$3^5 = 243 = 242 + 1 = 22 \times 11 + 1$, and so, $3^5 \mod 11 = 1$.
Then, $(3^5)^{10} \equiv 1 \mod 11$, and so, $3^{50} \equiv 1 \mod 11$. Then, $3^{51} \equiv 3 \mod 11$.
Thus, $6^{102} \mod 11 = 3$.

(9) (Exercise 2, Page 106) Today is Saturday. What day of the week will it be in 3724 days? What day of the week will it be in 365 days?

Solution: Since we are looking for days of the week, we are doing modular arithmetic with the number 7.

Note, 3724 is divisible by 7. In fact, $3724 = 532 \times 7$. Therefore, if today is Saturday, it will still be Saturday 3724 days later.
Next, $365 = 364 + 1 = 52 \times 7 + 1$, and so after 365 days, the day will be Saturday + 1, which is Sunday.