# SOLUTIONS TO HOMEWORK 3 - MATH 170, SUMMER SESSION I (2012)

(1) Let $\mathcal{U}$ be a universal set and $A \subset \mathcal{U}$. Using rules that you have learnt in class, simplify the following set theoretic expressions:

(a) $((A^c \cup A) - A) \cap A$.

<u>*Solution*</u>:

$$\begin{aligned} A^c \cup A &= \mathcal{U}, \\ \implies (A^c \cup A) - A &= \mathcal{U} - A = A^c, \\ \implies ((A^c \cup A) - A) \cap A &= A^c \cap A = \phi. \end{aligned}$$

(b) $((A \triangle A^c) \cap (A^c - \mathcal{U})$.

<u>*Solution*</u>:

$$\begin{aligned} A \triangle A^c &= (A - A^c) \cup (A^c - A) = A \cup A^c = \mathcal{U}. \\ \text{Also, } A^c - \mathcal{U} &= \phi. \\ \text{So, } (A \triangle A^c) \cap (A^c - \mathcal{U}) &= \mathcal{U} \cap \phi = \phi. \end{aligned}$$

(2) Suppose you have a tetrahedron (which is a four sided symmetric object), with each of its four sides numbered 1, 2, 3, 4, and a decahedron (which is a ten sided symmetric object), and you toss both of them and note down the number that comes up on each.

(a) Write down the universal set of this experiment.

<u>*Solution*</u>:
$\Omega = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9), (2,10),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8), (3,9), (3,10),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (4,7), (4,8), (4,9), (4,10)\}.$

(b) Express the event of getting a sum of 12 when you add up the two numbers in the tosses.

<u>*Solution*</u>:

$E = \{(2,10),(3,9),(4,8)\}$.

(c) Write down the probability of that event.

Solution:
$P(E) = \frac{n(E)}{n(\Omega)} = \frac{3}{40}$.

(d) Express the event of getting a product of 12 when you add up the two numbers in the tosses.

Solution:
$F = \{(2,6),(3,4),(4,3)\}$.

(e) Write down the probability of that event.

Solution:
$P(F) = \frac{n(F)}{n(\Omega)} = \frac{3}{40}$.

(3) A friend of yours has a coin that shows heads or tails. He tosses it once, notes down the outcome, then tosses it the second time, notes down the outcome, then the third time, and so on. At some point, you ask him when he intends to stop, to which he replies that he would stop when the coin gives him a head as its outcome for the third time.

(a) Find the probability that he stops tossing the coin after 5 tosses.

Solution: The total number of outcomes in 5 tosses is $2^5$. Now, he will stop after the fifth toss if the fifth toss gives him his third head. If we call this event $E$, then,

$$E = \{(H,H,T,T,H),(H,T,H,T,H),(H,T,T,H,H),\\ (T,T,H,H,H),(T,H,T,H,H),(T,H,H,T,H)\},$$

and so $n(E) = 6)$. Therefore,

$$P(E) = \frac{6}{2^5} = \frac{3}{16}.$$

(b) Find the probability that he does not stop in 4 tosses.

Solution: Let $F$ be the event that he does not stop in 4 tosses. Then, $F^c$ is the event that he DOES stop in 4 tosses. As before, the total number of outcomes in 4 tosses is $2^4$. Also,

$$F^c = \{(H, T, H, H), (H, H, T, H), (T, H, H, H)\},$$

which means $n(F^c) = 3$. Thus,

$$P(F^c) = \frac{3}{2^4} = \frac{3}{16},$$

and so,

$$P(F) = 1 - \frac{3}{16} = \frac{13}{16}.$$

(c) Can he stop after just two tosses?

*Solution*: No, he cannot. As the problem asks for him to stop at the third head, there needs to be at least three coin tosses for this to be true.

(4) Given a big box with 18 green and 12 red ball, say you draw three balls in the following manner. After each draw, you look at the color of the ball and replace it back in the box, after which you make the next draw.

(a) What is the probability that all the balls drawn are green?

*Solution*: Remember that this is drawing balls WITH REPLACEMENT. So individual draws are independent. The probability of getting a green ball in a single draw is $\frac{18}{30} = \frac{3}{5}$. So, the probability that all three balls drawn are green is $\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$.

(b) What is the probability that at least one of the balls is red?

*Solution*: Let $E$ be the event that at least one of the balls is red. Then $E^c$ is the event that none of the balls are red, i.e., all the balls are green, which is exactly what we computed in the previous part. So, $P(E^c) = \frac{27}{125}$. This gives, $P(E) = 1 - \frac{27}{125} = \frac{98}{125}$.

(c) What is the probability that exactly one ball is red?

*Solution*: The event "exactly one ball is red" is the union of three events $A, B$ and $C$, where $A$ is the event that the first ball is red, and the remaining two are green, $B$ is the event that the second ball is red, and the remaining two are green, and $C$ is the event that the third ball is red, and the first two are green. Now,

$$\begin{aligned}
P(A) &= \frac{12}{30} \times \frac{18}{30} \times \frac{18}{30} = \frac{18}{125}, \\
P(B) &= \frac{18}{30} \times \frac{12}{30} \times \frac{18}{30} = \frac{18}{125}, \\
P(C) &= \frac{18}{30} \times \frac{18}{30} \times \frac{12}{30} = \frac{18}{125}. \\
\implies P(A \cup B \cup C) &= \frac{18}{125} + \frac{18}{125} + \frac{18}{125} = \frac{54}{125}.
\end{aligned}$$

(d) If I were to change the question, wherein, you draw the three balls one by one without replacing any of them, in that case, how would you answer parts (a), (b) and (c) of this problem?

*Solution*: For part (a), the first green ball is drawn with probability $\frac{18}{30}$. Once the first ball is green, there will only be 17 green balls, and so the second ball is green with probability $\frac{17}{29}$. Once the first two balls turn out green, there will only be 16 green balls, and so the third ball is green with probability $\frac{16}{28}$. Therefore, the probability that all the three balls are green is equal to $\frac{18}{30} \times \frac{17}{29} \times \frac{16}{28} = \frac{204}{1015}$.

As in the with replacement case, the complement of the event "at least one ball is red" is the event "all balls are green". Thus, the probability that at least one ball is red is $1 - \frac{204}{1015} = \frac{811}{1015}$.

For part (c), we can again partition the event into $A, B$ and $C$, where $A$ is the event that the first ball is red, and the remaining two are green, $B$ is the event that the second ball is red, and the remaining two are green, and $C$ is the event that the third ball is red, and the first two are green. Now,

$$\begin{aligned}
P(A) &= \frac{12}{30} \times \frac{18}{29} \times \frac{17}{28} = \frac{102}{1015}, \\
P(B) &= \frac{18}{30} \times \frac{12}{29} \times \frac{17}{28} = \frac{102}{1015}, \\
P(C) &= \frac{18}{30} \times \frac{17}{29} \times \frac{12}{28} = \frac{102}{1015}. \\
\implies P(A \cup B \cup C) &= \frac{102}{1015} + \frac{102}{1015} + \frac{102}{1015} = \frac{306}{1015}.
\end{aligned}$$

(5) You now have two big boxes... Box 1 has 10 white and 10 black balls, and Box 2 has 14 white and 18 black balls. You choose a ball at random from Box 1, note its color and then drop it into Box 2. After that you choose a ball are random from

Box 2.

(a) What is the probability that both the balls turn out to be white?

<u>*Solution*</u>: For the first box, the probability of picking a white ball is $\frac{10}{20}$. Once a white ball had been picked, the second box now has 15 white and 18 black balls, and so the probability of picking a white ball from there is $\frac{15}{33}$. Hence, the probability that both balls are white is $\frac{10}{20} \times \frac{15}{33} = \frac{5}{22}$.

(b) What is the probability that both the balls turn out to be black?

<u>*Solution*</u>: For the first box, the probability of picking a black ball is $\frac{10}{20}$. Once a black ball had been picked, the second box now has 14 white and 19 black balls, and so the probability of picking a black ball from there is $\frac{19}{33}$. Hence, the probability that both balls are white is $\frac{10}{20} \times \frac{19}{33} = \frac{19}{66}$.

(c) What is the probability that the second ball is black? Is this higher than the probability that the second ball is white?

<u>*Solution*</u>: The event "the second ball is black" can be written as the union of two mutually exclusive events $A$ and $B$, where $A$ is the event that the first ball is white, and the second is black, and $B$ is the event that both are black. We already know from (b) that $P(B) = \frac{19}{66}$. Now, using the same reasoning as before, $P(A) = \frac{10}{20} \times \frac{18}{33} = \frac{3}{11}$.

Therefore, the probability that the second ball is black is $P(A \cup B) = P(A) + P(B) = \frac{19}{66} + \frac{3}{11} = \frac{37}{66}$. Note that, here we could add up the probabilities of $A$ and $B$ because they are mutually exclusive.

Next, The event "the second ball is white" can be written as the union of two mutually exclusive events $C$ and $D$, where $C$ is the event that the both balls are white, and $D$ is the event that the first ball is black and the second is white. We already know from (a) that $P(C) = \frac{5}{22}$. Now, using the same reasoning as before, $P(D) = \frac{10}{20} \times \frac{14}{33} = \frac{7}{33}$.

Therefore, the probability that the second ball is white is $P(C \cup D) = P(C) + P(D) = \frac{7}{33} + \frac{5}{22} = \frac{29}{66}$. Again, here we could add up the probabilities of $C$ and $D$ because they are mutually exclusive.

So, we see that the second ball has a greater chance of being black than being white.