MATH 371 FINAL - DUE WEDNESDAY, APRIL 30

1 a) [10] Let R be an integral domain with field of fractions F, and let $p(x)$ be a monic polynomial in $R[x]$. Assume that $p(x) = a(x)b(x)$ where $a(x), b(x)$ are monic polynomials in $F[x]$ of smaller degree than $p(x)$. Prove that if $a(x) \notin R[x]$, then R is not a Unique Factorization Domain.

b) [5] Show that $\mathbb{Z}[2\sqrt{2}]$ is not a UFD.

2 [10] Prove that if $f(x)$ and $g(x)$ are polynomials with rational coefficients whose product $f(x)g(x)$ has integer coefficients, then the product of any coefficient of $g(x)$ with any coefficient of $f(x)$ is an integer.

3 [10] Factor $x^{27} - x$ completely in $\mathbb{F}_3$. Prove that your factorization is irreducible.

4 [10] Let F be a field and $\alpha \in K$ for some extension K of F. Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$.

5 [10] Let $\theta = 1 + \sqrt[3]{2} + \sqrt[3]{4}$. Compute the irreducible polynomial of θ over $\mathbb{Q}$ and $[\mathbb{Q}(\theta) : \mathbb{Q}]$.

6 [10] Determine the Galois group of the splitting field of the polynomial $(x^2 - 2)(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$. Determine all the subfields of the splitting field of this polynomial.

7 a)[5] Determine the splitting field K of the polynomial $x^8 - 2$ over $\mathbb{Q}$.

b)[10] Compute $Gal(K/\mathbb{Q}(\sqrt{2}))$, $Gal(K/\mathbb{Q}(i))$, $Gal(K/\mathbb{Q}(\sqrt{-2}))$. In each of these cases, give an explicit isomorphism between the Galois group and a well-known group.