PRACTICE QUESTIONS FOR MIDTERM 1

1. Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 3 & 2 & 2 \\ 0 & 1 & -2 & 0 \end{pmatrix}$$

a) Find the rank of A
b) Find the dimensions of and bases for *rowsp(A)*, *colsp(A)*, *nullsp(A)*
c) Determine whether the vector $(1, 0, -2, 3) \in$ *rowsp(A)*
d) Extend the basis for $colsp(A)$ to a basis for $\mathbb{R}^4$.

Solution: (a & b) Row-reducing, we get

$$R = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A therefore has rank 2. A basis for the rowspace is given by the first two rows of R. The pivot variables are x_1, x_2, which means that the first two columns of A form a basis for the column space. To get the nullspace, we solve the system

$$x_1 + 2x_2 + x_4 = 0$$

$$-x_2 + 2x_3 = 0$$

in terms of the non-pivot variables x_3, x_4. We get $x_2 = 2x_3$, $x_1 = -4x_3 - x_4$. The nullspace therefore has the form

$$(-4x_3 - x_4, 2x_3, x_3, x_4) = x_3(-4, 2, 1, 0) + x_4(-1, 0, 0, 1)$$

so $(-4, 2, 1, 0), (-1, 0, 0, 1)$ form a basis.
(c) Suppose $(1, 0, -2, 3) \in rowsp(A)$ - then there are constants a, b such that

$$(1, 0, -2, 3) = a(1, 2, 0, 1) + b(0, -1, 2, 0)$$

This forces $a = 1, b = 2$, but the third equation yields $2b = -2$, which means the system is inconsistent. Thus $(1, 0, -2, 3)$ is not in the $rowsp(A)$.

(d) We want to extend $(1,1,2,0),(2,1,3,1)$ to a basis for $\mathbb{R}^4$. Putting these as rows in a 2×4 matrix and row-reducing, we get

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}$$

It follows that adding $(0,0,1,0),(0,0,0,1)$ yields a basis for $\mathbb{R}^4$.

2. Determine whether

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

is non-singular. If so, find B^{-1}.

Solution: Row - reducing, we get

$$R = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

All three pivots are non-zero, so the matrix is non-singular.

3. Find a basis for the smallest subspace of $\mathbb{R}^6$ containing the two subspaces $W_1 = span\{(1,2,3,0,3,2),(0,0,1,2,1,0)\}$ and $W_2 = span\{(1,2,2,-2,2,2),(0,0,1,2,1,1)\}$.

Solution: The smallest subspace is spanned by the four vectors $(1,2,3,0,3,2),(0,0,1,2,1,0)(1,2,2,-2,2,2),(0,0,1,2,1,1)$. To find a basis, put these as rows of a 4×6 matrix A and row-reduce.

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 & 3 & 2 \\ 1 & 2 & 2 & -2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 2 & 3 & 0 & 3 & 2 \\ 0 & 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We exchange rows 3 and 4 to get the zero row at the bottom. A basis for the desired subspace is therefore given by the three non-zero rows of R.

4.) Determine whether the statements are true or false. If true, say why, if false, give a counterexample.

(1) Let A be a 3×5 matrix such that $(1,1,1,1,1), (0,1,1,1,1), (0,0,1,1,1)$ are in *nullsp(A)*. The rows of A are linearly independent.
Solution: F $null(A) \geq 3$, so by the rank-nullity theorem, $rk(A) \leq 2$. The three rows are therefore linearly dependent.

(2) With A as in 1,the equation $Ax = B$ has a solution for every $B \in \mathbb{R}^3$.
Solution: F $rk(A) \leq 2$, and we would need rank 3 to guarantee a solution for every B.

(3) For A as in 1, the solution to $Ax = B$ when it exists is unique.
Solution: F When $B = 0$, we are dealing with the nullspace, which is at least three-dimensional. The solution is therefore not unique.

(4) A 5×27 matrix can have 6 linearly independent columns.
Solution: F $rk(A) \leq min(m,n) = 5$.

(5) Suppose that A is an $n \times n$ matrix and $B \in \mathbb{R}^n$ a vector such that $Ax = B$ has infinitely many solutions. Then the columns of A are linearly dependent.
Solution: T If $Ax = B$ has infinitely many solutions, then $null(A) > 0$, and so $rk(A) < n$

(6) Suppose A is a 3×7 matrix such that $Ax = B$ is solvable for every $B \in \mathbb{R}^3$. Then A has 3 linearly independent rows.
Solution: T If $Ax = B$ is solvable for every B, then $rk(A) = 3$, so the rows are linearly independent.

(7) Suppose that A is a matrix such that *colsp(A)* and *nullsp(A)* are both 2-dimensional. Then A must be a 4×4 matrix.
Solution: F The rank-nullity theorem yields that A must have 4 columns. However, if we start with a matrix A satisfying this property, and add a zero row, it will still satisfy the property.

(8) Suppose that A is an $n \times n$ matrix such that $Ax = B$ is solvable for every $B \in \mathbb{R}^n$. Then for every $n \times n$ matrix C, there exists a matrix D such that $AD = C$.
Solution: T A is nonsingular, so let $D = A^{-1}C$.

(9) The nullspace of a 3×4 matrix cannot consist only of the zero vector.
Solution: T $rk(A) \leq 3$, so $null(A) \geq 1$.

(10) The nullspace of a 4×3 matrix cannot consist only of the zero vector.
Solution: F Any rank 3 4×3 matrix A will have this property.

(11) Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ be a set of vectors in $\mathbb{R}^4$ that spans $\mathbb{R}^4$. Every subset of four vectors of S spans $\mathbb{R}^4$.
Solution: F One of the v_i could be the zero vector.

(12) The set of all vectors of the form $(1, x, y) \subset \mathbb{R}^3$, where $x, y \in \mathbb{R}$, is a subspace of $\mathbb{R}^3$.
Solution: F It does not contain 0.

(13) Let u, v, w be elements of a vector space V, and let $a = u+v, b = w$. Then $span\{a, b\}$ is contained in, but not necessarily equal to $span\{u, v, w\}$.
Solution: T The span is obviously contained in that of $\{u, v, w\}$. If the latter are linearly indep, the spans will not be equal.

(14) The set of all vectors of the form $(x, x^2) \in \mathbb{R}^2$, as x ranges over the real numbers, forms a subspace of $\mathbb{R}^2$.
Solution: F $(1, 1) + (2, 4) = (3, 5)$ which is not in the subset.

(15) All vector spaces are finite-dimensional.
Solution: F Consider the vector space of polynomials.