PRACTICE SOLUTIONS FOR MIDTERM 2

1.) a) Under the reflection, $(1, 0)$ goes to $(0, 1)$, and $(0, 1)$ to $(1, 0)$. The matrix is therefore

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

b) Projecting the vector $(1, 0)$ onto the line $y = x$ yields $(\frac{1}{2}, \frac{1}{2})$, and projecting $(0, 1)$ yields the same vector. The matrix is therefore

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

2.) Rotating the hyperbola by $\pi/4$ yields a hyperbola with asymptotes the x' and y' axes (I use x', y' for the rotated coordinates, and x, y for the original ones). Such a hyperbola has equation $x'y' = k$ where k is a constant. To find the constant, observe that the rotated hyperbola passes through the point $(\sqrt{2}, \sqrt{2})$. The equation of the rotated hyperbola is therefore $x'y' = 2$.

To find the equation of the original, we know that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We thus obtain

$$(\frac{x-y}{\sqrt{2}})(\frac{x+y}{\sqrt{2}}) = 2$$

or

$$x^2 - y^2 = 4$$

3.) $Q^{-1}AQ$ where

$$Q = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

4.) We diagonalize the matrix. The roots of the characteristic polynomial are $1 + 2i, 1 - 2i$, and a set of corresponding eigenvectors is $(2i, 1)$ and $(-2i, 1)$. Thus

$$Q^{-1}AQ = \begin{pmatrix} 1+21 & 0 \\ 0 & 1-2i \end{pmatrix}$$

where

$$Q = \begin{pmatrix} 2i & -2i \\ 1 & 1 \end{pmatrix}$$

So $A = QDQ^{-1}$, where D is the diagonal matrix on the right hand side above.

$$A^{20} = \begin{pmatrix} 2i & -2i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (1+2i)^{20} & 0 \\ 0 & (1-2i)^{20} \end{pmatrix} \begin{pmatrix} 2i & -2i \\ 1 & 1 \end{pmatrix}^{-1}$$

5.)

(1) True:

$$T(f_1 + f_2) = (f_1 + f_2)(1) = f_1(1) + f_2(1) = T(f_1) + T(f_2)$$
$$T(cf) = (cf)(1) = cf(1) = cT(f)$$

(2) False;

$$T(cf) = c\frac{df}{dx} + 1 \neq c(\frac{df}{dx} + 1)$$

(3) False: The transformation which projects to the x-axis transforms parallelograms to line segments.

(4) False: If T is linear, T(2,2) = 2 T(1,1)

(5) True: IF we can find a_1, a_2, a_3 not all zero such that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$$

then by linearity

$$a_1 T(v_1) + a_2 T(v_2) + a_3 T(v_3) = 0$$

(6) False: Choose your favorite pair of 2×2 matrices.

(7) True: If $Ax = b$ is not solvable for some b, then $rank(A) < n$ and so $det(A) = 0$

(8) True: $null(A) > 0$ so $rank(A) < n$.

(9) True: Performing row operations on the first n and last m rows separately, we can make A and B separately upper-triangular. At this point the determinant is the product of the diagonal terms.

(10) True

(11) True, the determinant is the product of the eigenvalues.

(12) False: The sum of eigenvectors with emphthe same eigenvalue is an eigenvector. For instance, suppose that v_1, v_2 are eigenvectors with eigenvalues 1 and 2 respectively. Then $T(v_1 + v_2) = v_1 + 2v_2$.