

HOMEWORK 4 - WED. MARCH 3

1. Prove that the number of non-isomorphic one-dimensional representations of a finite group G is $|G|/|[G, G]|$, where $[G, G]$ denotes the commutator subgroup.

2. Let W_i be an irreducible representation of G , V an arbitrary representation, and V_i the direct sum of all subrepresentations of V isomorphic to W_i . Let H_i be the space of linear maps $f : W_i \mapsto V$ such that $\rho_V(g)f = f\rho_{(W_i)}(g)$ for all $g \in G$.

a) Show that the dimension of H_i is equal to the number of times that W_i occurs in V .

b) Let G act on $H_i \otimes W_i$ through the tensor product of the trivial representation on H_i and the given representation on W_i . Show that the map:

$$F : H_i \otimes W_i \mapsto V_i$$

defined by

$$F(\Sigma T_\alpha \otimes w_\alpha) \mapsto \Sigma T_\alpha(w_\alpha)$$

is an isomorphism of $H_i \otimes W_i$ onto V_i .

c) Show that decomposing V_i into a direct sum of W_i 's is the same as choosing a basis of H_i .

d) Deduce that if V is an irreducible representation of G , then $V \oplus V$ has infinitely many direct sum decompositions into two copies of V .

3. Let χ be the character of 2-dimensional representation of G , and let x be an element of order 2 in G . Prove that $\chi(x) = 2, 0, -2$. Generalize this to n -dimensional representations.

4. Let χ be an irreducible character of G . Prove that for every element g in $Z(G)$, the center of G , we have $\chi(g) = \epsilon\chi(1)$, where ϵ is a root of unity.

5. Let χ be the character of some representation ρ of G . Prove that

a) If $\chi(g) = \chi(1)$ then $g \in \text{Ker}(\rho)$.

b) If $|\chi(g)| = \chi(1)$ and ρ is faithful, then $g \in Z(G)$ (the center of G).

6 a) Show that S_6 has an irreducible character of dimension 5.

b) Calculate the character table of A_5 .