

MIDTERM - DUE WEDNESDAY, MARCH 31

1. Suppose I is a principal ideal of the integral domain R . Prove that the R -module $I \otimes_R I$ has no non-zero torsion elements (i.e. $rm = 0$ with $r \neq 0$ implies $m = 0$).

2. Prove that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

3. Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \times \mathbb{C}$ as rings.

4. Calculate the character table of $S_3 \times S_3$

Let G be a finite group and $H \leq G$ a subgroup. Let V be a representation of H . We define the induced representation of G by

$$\text{Ind}_H^G(V) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$$

(This is a left $\mathbb{C}[G]$ -module and therefore a representation of G). If ρ is a character of H , denote by $\text{Ind}_H^G(\rho)$ the character of the induced representation.

5. If ρ is the character of V , show that

$$\text{Ind}_H^G(\rho)(g) = \sum_{i=1}^k \rho(g_i^{-1} g g_i)$$

where g_1, \dots, g_k are representatives for the left cosets of H in G , and $\rho(g_i^{-1} g g_i)$ is defined to be 0 if $g_i^{-1} g g_i$ is not in H .

6. Let $H \leq G$ as above, and let ρ be a character of H , and χ a character of G . Prove the following formula, called the Frobenius reciprocity formula:

$$\langle \chi|_H, \rho \rangle_H = \langle \chi, \text{Ind}_H^G(\rho) \rangle_G$$

where $\chi|_H$ denotes the restriction of χ to H .

7. Suppose that $G = H \times K$. Let V be a representation of H . Prove that $\text{Ind}_H^G(V) = V \otimes R_K$, where R_K denotes the regular representation of K .

8. a) Let \mathbb{F}_p denote the finite field with p elements. Suppose that $[K : \mathbb{F}_p] = n$, prove that K has p^n elements.

b) Prove that $x^5 - ax - 1 \in \mathbb{Z}[x]$ is irreducible unless $a = 0, 2, -1$.