SELECTED SOLUTIONS TO PS 4

Problem 1 σ_r is a multiplicative function. Thus if $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime factorization, then $\sigma_r(n) = \prod_{n=1}^k \sigma_r(p_n^{a_n})$. Now

$$\sigma_r(p_n^{a_n}) = 1 + p_n^r + p_n^{2r} + \cdots p_n^{a_n r} = \frac{p^{(a_n+1)r} - 1}{p_n^r - 1}$$

Problem 2 a) The largest prime p that can divide n is 7, since otherwise $\phi(n) \geq p - 1 > 6$. Thus the only primes that can divide n are $2, 3, 5, 7$. Furthermore, if $5|n$ then $4|\phi(n)$, so the list is down to $2, 3, 7$. Suppose that $n = 2^a 3^b 7^c$. Then $a \leq 2$ (otherwise $4|\phi(n)$), $b \leq 2$, and $c \leq 1$. Furthermore, $a = 2$ is impossible since any non-zero choice of b, c results in an n such that $4|\phi(n)$. Finally, if $b = 2$, then $c = 0$. We are thus down to 10 choices, of which $n = 9, 7, 14, 18$ work.

Part b) is handled similarly.

Problem 3 a) Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime power factorization of n. Since Λ is non-zero only for prime powers, we have

$$\begin{aligned}
\sum_{d|n} \Lambda(n) &= \sum_{n=1}^{k} \sum_{r=1}^{a_k} \Lambda(p_k^r) \\
&= \sum_{n=1}^{k} \sum_{r=1}^{a_k} \log(p_k) \\
&= \sum_{n=1}^{k} a_k \log(p_k) \\
&= \log(n)
\end{aligned}$$

b) Mobius inversion yields

$$\begin{aligned}\Lambda(n) &= \sum_{d|n} \mu(d)\log(n/d) \\ &= \sum_{d|n} \mu(d)(\log(n) - \log(d)) \\ &= \log(n)\sum_{d|n} \mu(d) - \sum_{d|n} \mu(d)\log(d)) \\ &= -\sum_{d|n} \mu(d)\log(d))\end{aligned}$$

where in the last step we have used $\sum_{d|n} \mu(d) = 0$ for $n > 1$.

Problem 4 a) It is easy to check that 2 is a primitive root $mod 11$ (just check that it has order 10). The other primitive roots will be 2^s where $1 \leq s < 10, gcd(s, 10) = 1$.

b) 3 is a primitive root $mod 17$. The others will be 3^s, $1 \leq s < 16, gcd(s, 16) = 1$.

Problem 5 a) $gcd(5, 12) = 1$, so there will be a unique solution. 2 is a primitive root $mod 13$, and $6 \equiv 2^5 mod 13$. Writing $x \equiv 2^k \mod 13$, and plugging into the equation $x^5 \equiv 6(mod 13)$, we obtain the equation $5k \equiv 5(mod 12)$, or $k \equiv 1(mod 12)$. Thus $x \equiv 2^1$ is the only solution.

b) $gcd(4, 12) = 4$ so there will be 4 solutions $mod 13$. $9 \equiv 2^8 mod 13$. Proceeding as in a), we get the equation $4k \equiv 8(mod 12)$, which is equivalent to $k \equiv 2(mod 3)$. $2, 5, 8, 11$ are the solutions $mod 12$. Thus $x \equiv 2^2, 2^5, 2^8, 2^{11}(mod 13)$ are the solutions.