Picard-Vessiot extensions over number fields

Diplomarbeit

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13. April 2005

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Abstract. We develop a Picard-Vessiot theory for arbitrary differential fields of charateristic 0. This theory is employed to show that every connected algebraic affine group scheme over K occurs as Galois group scheme of some differential equation over $K(t)$, where K is a subfield of the field $\mathbb{C}$ of complex numbers. In particular, this includes the interesting cases when K is number field and $K = \mathbb{R}$.

Contents

Introduction

Classical Galois theory arises from the desire to solve polynomial equations with coefficients in some field F. For this purpose, one considers the splitting field E/F which is generated by the roots of a given equation. The symmetries of these roots are naturally described by the group of automorphisms of E leaving F fixed. This finite group $\mathrm{Gal}(E/F)$ is called the Galois group of E/F. One can establish a correspondence between intermediate fields of E/F and subgroups of $\mathrm{Gal}(E/F)$, the so called Galois correspondence.

Differential Galois theory is a generalization of classical Galois theory. It deals with linear differential equations having coefficients in some differential field F with algebraically closed field of constants L of characteristic 0 (for example $F = L(t)$ where the derivation is given by $\frac{d}{dt}$). The analogue of the splitting field is the Picard-Vessiot extension E/F which is generated by the entries of a fundamental matrix for a differential equation. The group of differential automorphisms of E/F acts on the solution space of the equation and carries the structure of a linear algebraic group over L. This group is called the differential Galois group of E/F and we denote it by $\mathrm{Gal}_D(E/F)$. In further analogy, there is a correspondence between the intermediate fields of E/F and the closed subgroups of $\mathrm{Gal}_D(E/F)$.
For an exposition of Picard-Vessiot theory we refer to the book [PS03, Chapter 1]. A major drawback of this theory is that it is restricted to differential fields with algebraically closed field of constants. This assumption is made to ensure that the Picard-Vessiot extensions contain no new constants.

A natural question arising in both classical as well as differential Galois theory is the inverse problem: Which groups are Galois groups of some equation (polynomial or differential, respectively)?
In both theories this question is known to have a positive answer over the field $L(t)$ (with L algebraically closed of characteristic 0), in the sense that every potential group occurs as Galois group (see [MM99] for the classical, [Har02] for the differential case).

In the first chapter of this thesis, we develop a Picard-Vessiot theory over arbitrary differential fields of characteristic 0. In particular, we deal with non-algebraically closed fields of constants. Our approach to this problem is the technique of Galois descent. We sketch the basic ideas. Let K be the field of constants of a differential field F and suppose that K is not algebraically closed. We define Γ to be the absolute Galois group of K, i.e., the Galois group of $\overline{K}/K$ for an algebraic closure $\overline{K}$ of K. It can be shown that $\tilde{F} := F \otimes_K \overline{K}$ is a differential field with field of constants $\overline{K}$. Thus, we can use "classical" Picard-Vessiot theory over the field $\tilde{F}$. Next, given a Picard-Vessiot extension $\tilde{E}/\tilde{F}$ the problem is to find an extension E/F which has field of constants K and makes the diagram

$$\begin{tikzcd} & \tilde{E} \\ E \arrow[ur] & \\ & \tilde{F} \arrow[uu] \\ F \arrow[uu] \arrow[ur] & \end{tikzcd}$$

commutative and cartesian, where the latter means that the natural map $E \otimes_K \overline{K} \to \tilde{E}$ is an isomorphism. Of course, we want E/F to be generated by some fundamental system of a differential equation over F. In fact, the existence of such a field E comes along with the existence of an action of the Galois group Γ on $\tilde{E}$ extending the natural action on $\tilde{F}$. Namely, we define the field E to be the invariants in $\tilde{E}$ under such an action. By construction, the field of constants of E will be K. We say that the field $\tilde{E}$ descends to E along the Galois extension $\overline{K}/K$. There may exist different Galois actions on $\tilde{E}$, in which case E is not unique. If E'/F is an extension of differential fields such that $E' \otimes_K \overline{K}$ is isomorphic to $E \otimes_K \overline{K}$ over $\tilde{F}$, we call E' an $\overline{K}/K$-form of E.

We construct this descent theory in Chapter 1 of the thesis. Several natural problems arise in this new context:

- *Given a differential equation defined by a matrix $A \in \mathrm{M}_n(\tilde{F})$ how can we decide whether the corresponding Picard-Vessiot extension $\tilde{E}/\tilde{F}$ descends along $\overline{K}/K$?*
  We give a sufficient criterion which states that if the equation $\partial - A$ is equivalent to all its conjugates then $\tilde{E}$ descends.

- *Is there a nice algebraic object that describes the automorphisms of E/F and $\tilde{E}/\tilde{F}$ simultaneously?*
  We use the language of group schemes in order to describe the symmetry groups of the Picard-Vessiot extensions. When working over non algebraically closed fields it may happen that the K-valued points of the scheme do not determine the full structure of the scheme. Therefore, we define an algebraic affine group scheme $\mathcal{G}$ over K which we call the Galois group scheme of E/F having the properties

$$\mathcal{G}(K) = \mathrm{Aut}_D(E/F) \quad \text{and} \quad \mathcal{G}(\overline{K}) = \mathrm{Aut}_D(\tilde{E}/\tilde{F}).$$

- *Classification of the $\overline{K}/K$-forms of E.*
  We classify the $\overline{K}/K$-forms of E by means of the Galois cohomology $\mathrm{H}^1(\Gamma, \mathcal{G}(\overline{K}))$.

- *Given an $\overline{K}/K$-form E'/F of E/F, how are the Galois group schemes of E/F and E'/F related?*
  Suppose that $\mathcal{G}$ is the Galois group scheme of E/F. We show that the Galois group scheme of E'/F is an inner form of $\mathcal{G}$. This yields a classification of all Galois group schemes realized by $\overline{K}/K$-forms of E via the Galois cohomology set $\mathrm{Inn}_*(\mathrm{H}^1(\Gamma, \mathcal{G}(\overline{K})))$.

- *Which classical results "descend" to K?*
  The descent technique makes it possible to transport results to our new situation. We give a proof of a Galois correspondence and an alternative proof of the Torsor theorem which immediately generalizes to our situation. This theorem constitutes a nice geometric interpretation of the theory. As may be seen in the examples, new phenomena occur, e.g., a $\mathcal{G}$-torsor over $K(t)$ needs not to be trivial even for a connected group scheme $\mathcal{G}$ (as is implied over $\overline{K}(t)$ by the Theorem of Springer-Steinberg).

At the end of the first chapter we give some examples that illustrate the descent and classification results.

An application of the above Picard-Vessiot theory is given in Chapter 2 where we give a positive answer to the inverse problem of differential Galois theory over the field $K(t)$ for connected groups, where K is a subfield of the field $\mathbb{C}$ of complex numbers. Every connected algebraic affine group scheme over K occurs as Galois group scheme of a differential equation over $K(t)$.
This includes the case when K is a number field and the case $K = \mathbb{R}$ which is nice in view of examples. The main tool will be the notion of K-effectiveness. A Picard-Vessiot extension $E/K(t)$ will be called K-effective if it is induced by a matrix A which is a $K(t)$-rational point in the Lie algebra of the Galois group scheme of $E/K(t)$. This notion only makes sense for connected group schemes since the Lie algebra only carries information about the connected component of a group scheme. In the K-effective case we have an exact control over the realized group scheme.
The connected inverse problem over $\overline{K}(t)$ was solved in a constructive way by C. Mitschi and M. Singer [MS96]. We mimic their proof for reductive groups and show that descent along $\overline{K}/K$ is possible.
The generalization of the proof in the semi-simple case is quite straight forward. Some more work has to be done in order to realize K-tori. This is due to the fact that the K-isomorphism classes of n-dimensional K-tori are in bijection with the set $\mathrm{H}^1(\overline{K}/K, \mathrm{GL}_n(\mathbb{Z}))$. Thus K-tori occur in a great variety. Since we make use of the exponential map, we have to assume that K is embeddable into $\mathbb{C}$.
The realization of reductive group schemes then generalizes to our situation without complications.
Next, we will use the technique of embedding problems and some results of T. Oberlies [Obe03] in order to solve the general connected case. The main work lies in solving embedding problems with unipotent abelian kernel. Here we employ an upper bound criterion for the Galois group which is proved using analytic methods.
It is an interesting question whether the assumption on K to be a subfield of $\mathbb{C}$ can be disposed.

Chapter 1

Picard-Vessiot theory over non algebraically closed fields of constants

In this chapter, we develop a theory of Galois descent for Picard-Vessiot extensions. For an exposure of the classical Picard-Vessiot theory (i.e., with algebraically closed field of constants) we suggest the book [PS03]. The concept of descent was developed in a very general context by A. Grothendieck [Gro60] and we specialize these ideas to our setting. The descent theory provides a nice method for controlling constants when looking at Picard-Vessiot extensions over differential fields whose field of constants is not algebraically closed.

1.1 Galois descent for differential rings

We start by developing a descent theory for differential rings. The categorial setting is the following.

Definition 1.1. *Let* DRing *be the category of differential rings. The objects are pairs* $R = (R, \partial)$*, where* R *is a commutative ring with unit and* ∂ *is a derivation on* R*. The morphisms are ring homomorphisms* $f : R \longrightarrow S$ *with* $f \circ \partial = \partial \circ f$ *and the set of these morphisms is denoted by* $\mathrm{Hom}_{\mathsf{DRing}}(R, S)$.

In order to develop a descent theory in this category, we need the existence of fibred coproducts.

Proposition 1.2. *Fibred coproducts exist in* DRing.

Proof. Suppose $f : R \longrightarrow S$, $g : R \longrightarrow T$ are morphisms in DRing. We define the differential ring $S \otimes_R T$ with derivation given by extending

$$\partial(s \otimes t) := \partial(s) \otimes t + s \otimes \partial(t)$$

additively to $S \otimes_R T$. Note that this derivation is well defined, since we have

$$\begin{aligned}\partial(rs \otimes t) &= \partial(r)s \otimes t + r\partial(s) \otimes t + rs \otimes \partial(t)\\ &= s \otimes \partial(r)t + \partial(s) \otimes rt + s \otimes r\partial(t) = \partial(s \otimes rt)\end{aligned}$$

for all $r \in R$, $s \in S$ and $t \in T$.
Let $h : S \longrightarrow X$ and $j : T \longrightarrow X$ be morphisms in DRing with $h \circ f = j \circ g$. Forgetting the derivations, we know that $S \otimes_R T$ is the fibred coproduct in the category of commutative rings with unit. This implies the existence of a homomorphism of rings $\varphi = h \otimes j$ which makes the diagram

$$\begin{tikzcd}
 & & X \\
S \arrow[r, "\imath_S"] \arrow[urr, bend left, "h"] & S \otimes_R T \arrow[ur, "\varphi"] & \\
R \arrow[u, "f"] \arrow[r, "g"'] & T \arrow[u, "\imath_T"] \arrow[uur, bend right, "j"'] &
\end{tikzcd}$$

commutative where $\imath_S$ and $\imath_T$ denote the canonical inclusions. Clearly φ respects the differential structures by construction of the derivation on $S \otimes_R T$. $\square$

Definition 1.3. *Given a differential ring $F \in \mathsf{DRing}$ we define DRing_F to be the* **category of differential rings over** *F. The objects are morphisms $F \longrightarrow R$ in DRing and the morphisms are commutative diagrams*

$$\begin{tikzcd}
R \arrow[rr] & & S \\
 & F. \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

In other words, the objects of DRing carry an additional F-algebra structure and we require the morphisms to respect this structure.

If we provide a field K with the trivial derivation $\partial = 0$, we obtain an object in DRing. Using the terminology of Definition 1.3 we can define the category DRing_K. When looking at Picard-Vessiot rings, the field K will be the field of constants.

For the rest of this section, K denotes a field of characteristic 0 and L/K is a Galois extension. Let Γ denote the Galois group of L/K. It is a profinite group which carries a topology, namely the so called Krull topology. A basis of the open sets is given by the subsets of the form σN where $\sigma \in \Gamma$ and N is a normal subgroup of finite index in Γ.

In this situation we have a **base change functor**

$$\begin{aligned}
- \otimes_K L : \mathsf{DRing}_K \longrightarrow \mathsf{DRing}_L, \qquad & R & \mapsto R_L := R \otimes_K L \\
& (f : R \longrightarrow S) & \mapsto (f \otimes \mathrm{id}_L : R_L \longrightarrow S_L).
\end{aligned}$$

Note that $f \otimes_K \mathrm{id}_L$ is indeed a morphism in DRing_L. We write f_L short hand for $f \otimes_K L$. Now we have the terminology to formulate the descent question: Which objects of DRing_L do we obtain by applying the base change functor to an object in DRing_K? The theory of Galois descent will characterize these objects in terms of the Galois group Γ.

Definition 1.4. *Let $R \in \mathsf{DRing}$. We define a* Γ**-action on** R *to be a homomorphism of groups*

$$\rho : \Gamma \longrightarrow \mathrm{Aut}_{\mathsf{DRing}}(R).$$

We say that the Γ-action is **continuous** *if it has the additional property*

$$R = \bigcup_{\Delta} R^{\Delta} (= \varinjlim_{\Delta} R^{\Delta})$$

where Δ runs over the set of open subgroups of Γ. Here R^{Δ} denotes the set of elements of R which are invariant under the action of Δ via ρ. If no confusion can arise, we write $\sigma(r)$ instead of $\rho(\sigma)(r)$ for $\sigma \in \Gamma$ and $r \in R$. Let $R, R' \in \mathsf{DRing}$ and let $f : R \longrightarrow R'$ be a morphism. Given continuous Γ-actions on R and R', we say that f is Γ**-equivariant** *if*

$$f(\sigma(r)) = \sigma(f(r)), \quad \textit{for all } r \in R,\ \sigma \in \Gamma.$$

The continuity property in Definition 1.4 can be reformulated. It just means that every element of R is invariant under the action of some normal subgroup of Γ of finite index. The natural action of Γ on the field L is a continuous Γ-action. This is due to the fact that every element of L is contained in a finite Galois extension of K, thus, by Galois correspondence, is invariant under the action of some normal subgroup of Γ of finite index. From now on, we provide L with this action.
If L/K is a finite Galois extension then every Γ-action is continuous, since the trivial subgroup $1 \leq \Gamma$ has finite index.

Proposition 1.5. *Let $R \in \mathsf{DRing}_K$ with K-algebra structure $h : K \longrightarrow R$.*

(1) There exists a continuous Γ-action on R_L such that $h_L : L \longrightarrow R_L$ is Γ-equivariant.

(2) Let $S \in \mathsf{DRing}_K$ and let $f : R \longrightarrow S$ be a morphism in DRing_K. Then $f_L : R_L \longrightarrow S_L$ is equivariant with respect to the Γ-actions from (1).

Proof. For $\sigma \in \Gamma$ we define $\rho(\sigma)(\sum r \otimes l) := \sum r \otimes \sigma(l)$ for $\sum r \otimes l \in R_L$ and compute

$$\partial(\sigma(r \otimes l)) = \partial(r \otimes \sigma(l)) = \partial(r) \otimes \sigma(l) = \sigma(\partial(r \otimes l)).$$

Thus Γ acts via differential automorphisms.
The Γ-action on R_L is continuous. Indeed, if $r' = \sum_{i=1}^{n} r_i \otimes l_i \in R_L$ then each l_i is invariant under some open subgroup $\Delta_i \trianglelefteq \Gamma$. The finite intersection of the groups Δ_i is open and leaves r' invariant.
By construction the maps h_L and f_L are Γ-equivariant. □

Therefore, the objects in DRing_L which arise from objects in DRing_K via base change are equipped with a natural Γ-action. Next we show that this property already characterizes those objects.

Definition 1.6. *A* **descent datum** *to K in DRing_L is a pair $(\tilde{R}, \rho)$ in which $\tilde{R}$ is an object in DRing_L and ρ is a continuous Γ-action on $\tilde{R}$, such that the structure map $L \longrightarrow \tilde{R}$ of $\tilde{R}$ is Γ-equivariant.*
We define the category $\mathsf{DRing}_L^{\Gamma}$, having the descent data as objects. The morphisms are the morphisms in DRing_L which are Γ-equivariant with respect to the given Γ-actions.

Given an object R in DRing_K, we obtain an object in DRing_L^Γ by applying the base change functor and equipping R_L with the natural Γ-action from Proposition 1.5. Still by Proposition 1.5, we know that a morphism in DRing_K yields, by base change, a Γ-equivariant morphism in DRing_L.
Consequently the base change functor induces a functor

$$\mathcal{B} : \mathsf{DRing}_K \longrightarrow \mathsf{DRing}_L^\Gamma.$$

Our answer to the descent question is given by showing that $\mathcal{B}$ is an equivalence of categories. For this, we have to show that $\mathcal{B}$ is fully faithful (descent of morphisms) and that every object in DRing_L^Γ is isomorphic to some object of the form $\mathcal{B}(R)$ with $R \in \mathsf{DRing}_K$ (descent of objects). We start by proving that $\mathcal{B}$ is fully faithful.

Lemma 1.7. $(R_L)^\Gamma = R$.

Proof. The nontrivial inclusion is $(R_L)^\Gamma \subset R$. For this, let $\{r_i | i \in I\}$ be a K-basis of R, where I is an index set. Then $\{r_i \otimes 1 | i \in I\}$ is an L-basis of R_L. Thus, we can represent every element of R_L by a sum

$$\sum_i r_i \otimes l_i,$$

where $l_i \in L$ and i runs over a finite subset of I. If such an element lies in $(R_L)^\Gamma$ we have

$$\sum_i r_i \otimes l_i = \sum_i r_i \otimes \sigma(l_i)$$

for all $\sigma \in \Gamma$. But this means that $l_i \in K$ and the assertion follows. □

Theorem 1.8 (Descent of morphisms). *Let R, $S \in \mathsf{DRing}_K$. Then the additive map*

$$\mathcal{B} : \ \mathrm{Hom}_{\mathsf{DRing}_K}(R, S) \longrightarrow \mathrm{Hom}_{\mathsf{DRing}_L^\Gamma}((R_L, \rho_R), (S_L, \rho_S)), \quad f \mapsto f_L$$

is a bijection.

Proof.

- *$\mathcal{B}$ is injective*
  The morphism $\imath_S : S \longrightarrow S \otimes_K L$, $s \mapsto s \otimes 1$ is injective. By definition of $\mathcal{B}(f) = f_L$ we have $\imath_S \circ f = f_L \circ \imath_R$ so $f_L = 0$ implies $f = 0$.

- *$\mathcal{B}$ is surjective*
  We know that $R_L^\Gamma = R$ thanks to Lemma 1.7. Let $h \in \mathrm{Hom}_{\mathsf{DRing}_L^\Gamma}((R_L, \rho_R), (S_L, \rho_S))$. The Γ-equivariance of h implies that

  $$h(R) = h((R_L)^\Gamma) \subset h(R_L)^\Gamma \subset (S_L)^\Gamma = S.$$

  In other words, we can restrict h to obtain a morphism $h|_R \in \mathrm{Hom}_{\mathsf{DRing}_K}(R, S)$. We check that h and $h|_R \otimes L$ agree on the L-basis $\{r_i \otimes 1 | i \in I\}$, thus, they must agree on all of R_L due to the L-linearity of both maps.

□

Lemma 1.9. *Let H be a monoid and L a field. Then pairwise distinct characters of H in L, i.e., homomorphisms $H \longrightarrow L^*$, are linearly independent over L.*

Proof. [Lan84, Chapter VIII, §4, Theorem 4.1] □

Corollary 1.10. *Let L/K be a finite Galois extension with $\Gamma = \{\sigma_1, \ldots, \sigma_n\}$ and let $l_1, \ldots, l_n$ be a K-basis of L. Then the matrix*

$$\begin{pmatrix} \sigma_1(l_1) & \cdots & \sigma_1(l_n) \\ \vdots & & \vdots \\ \sigma_n(l_1) & \cdots & \sigma_n(l_n) \end{pmatrix}$$

is invertible.

Proof. We consider $\sigma_i : L^* \longrightarrow L^*$ as characters of L^* in L. Let $\lambda_i \in L$ with

$$\sum_{i=1}^{n} \lambda_i \begin{pmatrix} \sigma_i(l_1) \\ \vdots \\ \sigma_i(l_n) \end{pmatrix} = 0.$$

Then for all $1 \leq j \leq n$ we have

$$(\sum_{i=1}^{n} \lambda_i \sigma_i)(l_j) = 0$$

and thus $\sum \lambda_i \sigma_i = 0$ on L, because $\{l_j\}$ is a K-basis of L. The assertion follows from Lemma 1.9. □

Equipped with the last corollary, we are able to treat the descent of objects in DRing_L^Γ. First, we deal with a finite Galois extension L/K. The general case will be reduced to this case.

Theorem 1.11 (Descent of objects). *Let L/K be a finite Galois extension and $(\tilde{R}, \rho) \in \mathsf{DRing}_L^\Gamma$. Then there exists a ring $R \in \mathsf{DRing}_K$ and a morphism $f : R \longrightarrow \tilde{R}$ such that*

$$\begin{tikzcd} & \tilde{R} \\ R \arrow[ur, "f"] & \\ & L \arrow[uu, "g"'] \\ K \arrow[uu] \arrow[ur] & \end{tikzcd}$$

is a cartesian diagram, i.e., the diagram is commutative and the natural morphism $f \otimes g : R \otimes_K L \longrightarrow \tilde{R}$ is an isomorphism in DRing_L^Γ.

Proof. Let $\Gamma = \{\sigma_1, \ldots, \sigma_n\}$. We define R to be the ring of invariants of $\tilde{R}$ under the given Γ-action ρ. Because Γ acts via differential automorphisms, R is stable under the derivation of $\tilde{R}$. Due to its Γ-equivariance, the map $g : L \longrightarrow \tilde{R}$ induces a morphism $K \longrightarrow R$ by restriction. Thus we obtain an object R in DRing_K and it remains to show that the L-linear map $f \otimes g : R \otimes_K L \longrightarrow \tilde{R}, \sum r \otimes l \mapsto \sum rl$ is an isomorphism.

- *$f \otimes g$ is surjective.*
  We choose a K-basis $\{l_1, \dots, l_n\}$ of L and let $\tilde{r} \in \tilde{R}$. For every $1 \le j \le n$
  $$\sum_{i=1}^{n} \sigma_i(\tilde{r} l_j) = \sum_{i=1}^{n} \sigma_i(\tilde{r})\sigma_i(l_j)$$
  is Γ-invariant. Using Corollary 1.10 we see that the matrix $(\sigma_i(l_j))$ is invertible thus $\sigma(\tilde{r})$ is an L-linear combination of invariants for every $\sigma \in \Gamma$. The assertion follows if we consider the case $\sigma = 1$.

- *$f \otimes g$ is injective.*
  Let $\sum_{j=1}^{n} r_j \otimes l_j \in R \otimes_K L$. Suppose $\sum r_j l_j = 0$ in $\tilde{R}$ then $\sum r_j \sigma(l_j) = 0$ for all $\sigma \in \Gamma$. So we have
  $$\begin{pmatrix} \sigma_1(l_1) & \dots & \sigma_1(l_n) \\ \vdots & & \vdots \\ \sigma_n(l_1) & \dots & \sigma_n(l_n) \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = 0$$
  and by applying Corollary 1.10 we conclude $r_j = 0$ for all j.

□

Next we prove the same assertion for infinite Galois extensions. Recall that in this case the Galois group is a profinite group which is equipped with the Krull topology.

Corollary 1.12. *Theorem 1.11 remains true for L/K infinite Galois.*

Proof. For every open normal subgroup Δ of Γ we may write $\tilde{R}^\Gamma = (\tilde{R}^\Delta)^{\Gamma/\Delta}$ and thus have
$$\tilde{R}^\Gamma \otimes_K L^\Delta \cong (\tilde{R}^\Delta)^{\Gamma/\Delta} \otimes_K L^\Delta \cong \tilde{R}^\Delta$$
where the last isomorphism is due to Theorem 1.11. We now take the direct limit over the open normal subgroups of Γ. By continuity of the Γ-action on $\tilde{R}$ and L we have
$$\varinjlim_{\Delta} \tilde{R}^\Delta = \tilde{R} \quad \text{and} \quad \varinjlim_{\Delta} L^\Delta = L.$$
The assertion follows because direct limits commute with tensor products [AM69, Chapter 2, Excercise 20]. □

1.2 Galois descent for Picard-Vessiot rings

In the classical theory over algebraically closed fields of constants, Picard-Vessiot extensions have no new constants without explicitly requiring it (see [PS03, Lemma 1.17]). Trying to naively extend the theory to non algebraically closed fields of constants, the Picard-Vessiot extensions will in general have new constants. Our approach to handle this problem is descent theory.
Throughout this section we use the following notation. K is a field of characteristic 0 and L is a Galois extension of K with Galois group Γ. As in the previous section we provide Γ with the Krull topology. F denotes a differential field whose field of constants is K. We have the following proposition.

Proposition 1.13. *$\tilde{F} := F \otimes_K L$ is a differential field and its field of constants equals L.*

Proof. [Sti93, III.6.1] □

Definition 1.14. *Let $A \in \mathrm{M}_n(F)$. A* **pseudo Picard-Vessiot ring** *over F for the differential equation $\partial - A$ is a differential ring $R \in \mathsf{DRing}_F$ such that*

(1) R is a simple differential ring, i.e., R contains no nontrivial differential ideals,

(2) there exists a fundamental solution matrix $Y \in \mathrm{GL}_n(R)$, i.e., $\partial(Y) = AY$,

(3) R is "generated by Y", i.e., $R = F[Y_{ij}, \det(Y)^{-1}]$,

(4) there are no new constants in the field of fractions of R, i.e., for $x \in \mathrm{Quot}(R)$ we have $\partial(x) = 0$ if and only if $x \in K$.

Let $G = \mathrm{Aut}_{\mathsf{DRing}_F}(R)$. If R has the additional property

(5) $R^G = F$

then we call R a **Picard-Vessiot ring** *over F for $\partial - A$.*

Note that since F is a field, a morphism $F \longrightarrow R$ in DRing is automatically injective, thus we can identify F with a differential subring of R.
Condition (4) makes sense, since condition (1) together with (3) implies that R is a domain.
In the classical definition of a Picard-Vessiot ring the conditions (4) and (5) are omitted. This is because they can be deduced from the other conditions if the field of constants is algebraically closed.
A (pseudo) Picard-Vessiot ring over F without reference to a special equation is meant to be a (pseudo) Picard-Vessiot ring for some equation.

Definition 1.15. *The field of fractions of a (pseudo) Picard-Vessiot ring is called a* **(pseudo) Picard-Vessiot field***.*

We define some new categories and compare them with the ones from the previous section.

Definition 1.16. *We denote by PPV_F the* **category of pseudo Picard-Vessiot rings over** *F. The morphisms are the morphisms in DRing_F, thus PPV_F is a full subcategory of DRing_F. We also define PV_F as the* **category of Picard-Vessiot rings over** *F and remark that PV_F is a full subcategory of PPV_F.*

We want to set up a descent theory for pseudo Picard-Vessiot rings. Given $R \in \mathsf{PPV}_F$ we also have $R \in \mathsf{DRing}_K$ since $F \in \mathsf{DRing}_K$. Thus we may apply the base change functor $- \otimes_K L$ to R and obtain an object $R_L \in \mathsf{DRing}_L$. Via base change of the structure map $F \longrightarrow R$ to $F_L \longrightarrow R_L$ we even have $R_L \in \mathsf{DRing}_{F_L}$. The next proposition shows that the pseudo Picard-Vessiot structure is stable under base change. To this end, we employ the following lemma which is a (trivial) generalization of [PS03, 1.4, Lemma 1.29].

Lemma 1.17. *Let M be a field and N/M an arbitrary field extension. Let G be a group acting on N via field automorphisms over M such that $N^G = M$. Let B be an M-Algebra and $B_N = B \otimes_M N$ with G-action on the first component. Then the map*

$$\psi : \mathcal{I}(B) \longrightarrow \mathcal{I}^G(B_N),\ I \mapsto I \otimes_M N$$

is an isomorphism of the lattice of ideals in B onto the lattice of G-stable ideals in B_N with inverse map

$$\psi^{-1} : \mathcal{I}^G(B_N) \longrightarrow \mathcal{I}(B),\ J \mapsto J \cap B.$$

Proof. The nontrivial assertion is that $J \subset (J \cap B) \otimes_M N$. Let $\{b_i | i \in \Upsilon\}$ with index set Υ be a M-basis of B. Every element $y \in J$ can be written as a sum

$$y = \sum_{i \in \Omega} b_i \otimes y_i,$$

where Ω is a finite subset of Υ and $y_i \in N$ for all $i \in \Omega$. We define the length $|y|$ of y to be the cardinality of Ω and prove the assertion by induction on the length of y.
The cases $|y| \in \{0, 1\}$ are clear, so let $|y| > 2$. We may suppose $y_i = 1$ for some i. In addition we suppose $y_j \in N \setminus M$ for some j because otherwise $y \in J \cap B$ and we would be finished. Then $|g(y) - y| < |y|$ for all $g \in G$ and thus, by induction, $g(y) - y \in (J \cap B) \otimes_M J$. Since $N^G = M$ there is some $g \in G$ such that $g(y_j) \neq y_j$, so the computation

$$\underbrace{g(y_j^{-1}y) - y_j^{-1}y}_{\in N \otimes_M (J \cap B)} = g(y_j^{-1})\ \underbrace{(g(y) - y)}_{\in N \otimes_M (J \cap B)} + \underbrace{(g(y_j^{-1}) - y_j^{-1})}_{\neq 0}\, y$$

completes the induction. □

Proposition 1.18. *Let R be a pseudo Picard-Vessiot ring over F. Then R_L is a pseudo Picard-Vessiot ring over F_L.*

Proof. We choose an algebraic closure $\overline{K}$ of K with $L \subset \overline{K}$ and first prove that $R_{\overline{K}}$ is a pseudo Picard-Vessiot ring. In the diagram

$$\begin{tikzcd}
\mathrm{Quot}(R) \arrow[r] & \mathrm{Quot}(R)_L \arrow[r] & \mathrm{Quot}(R)_{\overline{K}} \\
R \arrow[u] \arrow[r] & R_L \arrow[u] \arrow[r] & R_{\overline{K}} \arrow[u] \\
F \arrow[u] \arrow[r] & F_L \arrow[u] \arrow[r] & F_{\overline{K}} \arrow[u]
\end{tikzcd}$$

we have, by Proposition 1.13, that $\mathrm{Quot}(R)_{\overline{K}}$ is a field with field of constants $\overline{K}$. Since $\mathrm{Quot}(R)$ is generated over F by the entries of a fundamental matrix, $\mathrm{Quot}(R)_{\overline{K}}$ is

generated over $F_{\overline{K}}$ by the same elements. The classical Picard-Vessiot theory tells us that $\mathrm{Quot}(R)_{\overline{K}}$ is a Picard-Vessiot extension [PS03, p.16, Proposition 1.22]. Because $-\otimes_K \overline{K}$ is exact we have an injection $R_{\overline{K}} \hookrightarrow \mathrm{Quot}(R)_{\overline{K}}$. Thus $R_{\overline{K}}$ is a domain and it is clear that $\mathrm{Quot}(R_{\overline{K}}) = \mathrm{Quot}(R)_{\overline{K}}$.
We also know that

$$R = F[Y_{ij}, \det(Y)^{-1}]$$

where Y is a fundamental matrix for some equation over F and therefore

$$R_{\overline{K}} = F_{\overline{K}}[Y_{ij}, \det(Y)^{-1}].$$

We conclude that $R_{\overline{K}}$ is a Picard-Vessiot ring for the same equation considered as an equation over $F_{\overline{K}}$ ($R_{\overline{K}}$ is differentially finite and generated by Y [PS03, Corollary 1.38]). In particular, $R_{\overline{K}}$ is a simple differential ring (this is quite surprising, if we look at Lemma 1.17!).
Next we apply the descent theory of the previous section to our situation. The extension $\overline{K}/L$ is Galois and we denote the Galois group by Δ. We have

$$(R_L) \otimes_L \overline{K} \cong R_{\overline{K}}$$

so $R_L = (R_{\overline{K}})^{\Delta}$ where the action is the natural Δ-action on $R_{\overline{K}}$. The lattice of differential ideals in R_L is in 1-1 correspondence with the lattice of Δ-stable differential ideals in $R_{\overline{K}}$ by Lemma 1.17. This implies that R_L is a simple differential ring (we could also use the fact that $\overline{K}/L$ is faithfully flat but the lemma provides a nice overview and is need later anyway).
The other properties of a pseudo Picard-Vessiot ring are immediately checked to be fulfilled by R_L and the assertion follows. □

Definition 1.19. *Let R be a pseudo Picard-Vessiot ring over F. An element $r \in R$ is called* **differentially finite** *if there exist $a_1, \dots, a_k \in F$ such that*

$$\sum_{i=1}^{k} a_i \partial(r) = 0.$$

We generalize a result from classical theory.

Corollary 1.20. *Let $R \in \mathsf{PPV}_F$ with pseudo Picard-Vessiot field $E = \mathrm{Quot}(R)$. Then R consists exactly of the elements of E which are differentially finite.*

Proof. The statement is true in the classical case. From the proof of Proposition 1.18 we know that we have a commutative diagram

$$\begin{tikzcd} E \arrow[r] & \mathrm{Quot}(R_{\overline{K}}) \\ R \arrow[u] \arrow[r] & R_{\overline{K}} \arrow[u] \end{tikzcd}$$

which is cartesian, i.e., $\mathrm{Quot}(R_{\overline{K}}) = E \otimes_K \overline{K}$. Let $x \in E$ be differentially finite. The classical theory implies $x \in R_{\overline{K}}$ because $\mathrm{Quot}(R_{\overline{K}})$ is a Picard-Vessiot field with

Picard-Vessiot ring $R_{\overline{K}}$. But since x is invariant under the natural Galois action on $E \otimes_K \overline{K}$, resp. $R \otimes_K \overline{K}$, we conclude $x \in R$, using Lemma 1.7.
Using the fact that R is generated by a fundamental matrix (which consists of differentially finite elements) we see that all elements in R are differentially finite. □

Definition 1.21. *Let $\tilde{R}$ be a pseudo Picard-Vessiot ring over $\tilde{F}$. A pseudo Picard-Vessiot ring R over F is called a* K**-structure on** *$\tilde{R}$ if there exists a morphism $f : R \longrightarrow \tilde{R}$ in* DRing *such that the diagram*

$$\begin{tikzcd}
 & \tilde{R} \\
R \arrow[ur, "f"] & \\
 & \tilde{F} \arrow[uu] \\
F \arrow[uu] \arrow[ur] & \\
 & L \arrow[uu] \\
K \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

is cartesian, i.e., the natural morphism $R \otimes_K L \longrightarrow \tilde{R}$ is an isomorphism in DRing_L*.*
A pseudo Picard-Vessiot ring R' over F is called an L/K**-form** *of R if $R_L \cong R'_L$ in* DRing_L*.*

We note that a Γ-action on a pseudo Picard-Vessiot ring $\tilde{R}$ over $\tilde{F}$ extends to a unique Γ-action on the corresponding pseudo Picard-Vessiot field $\tilde{E} = \mathrm{Quot}(\tilde{R})$. Conversely a Γ-action on $\tilde{E}$ restricts to a Γ-action on $\tilde{R}$. That is because the elements in $\tilde{R}$ are characterized by being differentially finite (Corollary 1.20) and this property is preserved by differential automorphisms. It is clear that continouity is preserved under this correspondence. Thus a continuous Γ-action on $\tilde{R}$ is the same thing as a continuous Γ-action on $\tilde{E}$. In the following (in particular in the following theorem), we will use this fact without explicitly noting it.

For every normal open subgroup H of Γ the finite extension $(\tilde{F})^H/F$ is Galois due to [Lan84, VI, §1, Theorem 1.8]. Taking the direct limit, we conclude that $\tilde{F}/F$ is Galois. We remark that we could define the notion of an F-structure R on a pseudo Picard-Vessiot ring $\tilde{R}$ over $\tilde{F}$. This is the same thing as a K-structure since $\mathrm{Gal}(\tilde{F}/F) \cong \mathrm{Gal}(L/K)$ and $R \otimes_K L \cong R \otimes_F \tilde{F}$. We prefer the description by the fields of constants because it is more natural when it comes to exploring the relation between K-structures on pseudo Picard-Vessiot rings and K-structures on the corresponding Galois group schemes. Those schemes will be defined in the next section.

In analogy to Definition 1.6 we define the notion of a descent datum.

Definition 1.22. *A* **descent datum** *to K in $\mathsf{PPV}_{\tilde{F}}$ is a pair $(\tilde{R}, \rho)$ in which $\tilde{R}$ is an object in $\mathsf{PPV}_{\tilde{F}}$ and ρ is a continuous Γ-action on $\tilde{R}$, such that the structure map $\tilde{F} \longrightarrow \tilde{R}$ is Γ-equivariant (where $\tilde{F} = F \otimes_K L$ is provided with the natural Γ-action).*
We define the category $\mathsf{PPV}_{\tilde{F}}^{\Gamma}$, having the descent data as objects. The morphisms are the morphisms in $\mathsf{PPV}_{\tilde{F}}$ which are Γ-equivariant with respect to the given Γ-actions.

We translate Proposition 1.18 into this categorical language. It implies the existence of a base change functor

$$\mathcal{B} : \mathsf{PPV}_F \longrightarrow \mathsf{PPV}^{\Gamma}_{\tilde{F}}$$

which is defined in an obvious way (still in analogy to the base change functor for differential rings).
We proceed by showing that $\mathcal{B}$ is an equivalence of categories. Since the morphisms of Picard-Vessiot rings over F are just morphisms in the category DRing_F we conclude that $\mathcal{B}$ is fully faithful due to Theorem 1.8.
It remains to show that objects in $\mathsf{PPV}^{\Gamma}_{\tilde{F}}$ descend to objects in PPV_F.

Theorem 1.23. *The functor $\mathcal{B}$ is an equivalence of categories.*

Proof. By the above considerations it is enough to show that objects in $\mathsf{PPV}^{\Gamma}_{\tilde{F}}$ descend to objects in PPV_F. Let $(\tilde{R}, \rho) \in \mathsf{PPV}^{\Gamma}_{\tilde{F}}$ then, forgetting about the Picard-Vessiot structure, Corollary 1.12 and Theorem 1.8 imply that, by taking invariants under ρ, we obtain an object $R \in \mathsf{DRing}_K$ with $F \longrightarrow R$ in DRing such that the diagram

$$\begin{tikzcd}
 & \tilde{R} \\
R \arrow[ur, "f"] & \\
 & \tilde{F} \arrow[uu] \\
F \arrow[uu] \arrow[ur] & \\
 & L \arrow[uu] \\
K \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

is cartesian. It remains to prove that R is a pseudo Picard-Vessiot ring over F.

- *R is simple.*
  This is clear from Lemma 1.17.

- $\mathrm{Quot}(R)$ *contains no new constants.*
  We have $\mathrm{Quot}(R) \hookrightarrow \mathrm{Quot}(\tilde{R})$. Let $x \in \mathrm{Quot}(R)$ with $\partial(x) = 0$. Then $x \in K$ since $\tilde{R}$ is a pseudo Picard-Vessiot ring,which implies $x \in R$, and since x is invariant under Γ.

- *R is generated by the fundamental matrix of an equation with coefficients in F.*
  We define $E := \mathrm{Quot}(R)$ and $\tilde{E} := \mathrm{Quot}(\tilde{R})$. Let $\tilde{E}$ be generated by the entries of the fundamental solution matrix $Y \in \mathrm{GL}_n(\tilde{R})$ for the equation defined by $A \in \mathrm{M}_n(\tilde{F})$. Since the action of Γ on $\tilde{E}$ is continuous, the field $E(Y_{ij}, A_{ij})$ is invariant under an open normal subgroup Δ of Γ. Then by [Lan84, VI, §1, Theorem 1.8] the field $E' := \tilde{E}^{\Delta}$ is a finite Galois extension of E with Galois group $\Lambda = \Gamma/\Delta$. The same theorem implies that $F' := \tilde{F}^{\Delta}$ is Galois over F with the same Galois group Λ. Finally, we know that $K' := L^{\Delta}$ is Galois over K and the Galois group is (surprise) Λ. K' is the field of constants of F' and E'. The Γ-equivariance of $\tilde{F} \longrightarrow \tilde{R}$ implies the Γ-equivariance of $\tilde{F} \longrightarrow \tilde{E}$ which yields a

morphism $F' \longrightarrow E'$. If we also define $R' := R^{\Delta}$ we can draw the situation in a diagram.

$$\begin{tikzcd}
E \arrow[r] & E' \arrow[r] & \tilde{E} \\
R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[r] & \tilde{R} \arrow[u] \\
F \arrow[u] \arrow[r] & F' \arrow[u] \arrow[r] & \tilde{F} \arrow[u] \\
K \arrow[u] \arrow[r] & K' \arrow[u] \arrow[r] & L \arrow[u]
\end{tikzcd}$$

By construction, we have $A \in \mathrm{GL}_n(F')$ and $Y \in \mathrm{GL}_n(E')$. We choose a K-basis $B = (1, b_1, \ldots, b_m)$ of K' where m denotes the order of Λ. Then B is an F-basis of F' as well as an E-basis of E'. The elements of $F' \subset E'$ are exactly the elements having coordinates in F with respect to the basis B. Let

$$\mu : E' \longrightarrow \mathrm{M}_m(E)$$

be the map that sends each element $x \in E'$ to the representation matrix of the endomorphism "multiplication by x" of E' with respect to B. If we restrict μ to F', we obtain matrices with entries in F. It is clear that μ commutes with the derivation because we chose our basis to consist of constants. Thus the map μ is a homomorphism of differential rings.
We have

$$\partial(\mu(Y)) = \mu(\partial(Y)) = \mu(AY) = \mu(A)\mu(Y) \quad (\in \mathrm{M}_{mn}(E))$$

where μ is applied to every entry of the respective matrices. Furthermore, we have

$$1 = \mu(YY^{-1}) = \mu(Y)\mu(Y^{-1})$$

and so $\mu(Y) \in \mathrm{GL}_{mn}(E)$. By the above considerations we have $\mu(A) \in \mathrm{M}_{mn}(F)$. It is clear that the entries of $\mu(Y)$ generate E' over F' because all entries of Y are K'-linear combinations of elements of $\mu(Y)$ due to the fact that the first element of B is 1. So we have $F(\mu(Y)_{ij}) \otimes_K K' = E'$ and thus $E = F(\mu(Y)_{ij})$ by Lemma 1.7.
It remains to check that $R = F[\mu(Y)_{ij}, \det(\mu(Y))^{-1}]$. On behalf of better readability, we define $S := F[\mu(Y)_{ij}, \det(\mu(Y))^{-1}]$. By Lemma 1.7, it is enough to show that $S \otimes_K L \cong \tilde{R}$. For this we note that all $\mu(Y)_{ij}$ and $\det(\mu(Y))^{-1}$ are differentially finite, thus

$$S \otimes_K L \subset \tilde{R}$$

by Corollary 1.20. It is clear that $Y_{ij} \in S \otimes_K L$, again because of the element 1 in our basis, and $\det(Y)^{-1} \in S \otimes_K L$ by the formula

$$\det(Y)^{-1} = \det(\mu(Y))^{-1} \prod_{\substack{\sigma \in \Lambda \\ \sigma \neq \mathrm{id}}} \sigma(\det(Y))$$

which is proved in the following Lemma 1.24. We conclude that

$$S \otimes_K L \supset \tilde{R}$$

which proves our claim.

$\square$

Lemma 1.24. *With the notation of the proof of Theorem 1.23 we have*

$$\det(\mu(Y)) = \prod_{\sigma \in \Lambda} \sigma(\det(Y)).$$

Proof. For $y \in E'$, the minimal polynonial over E is given by

$$P_y(X) = \prod_{\sigma \in V} (X - \sigma(y))$$

where V is a system of representatives in Λ with respect to the relation

$$\sigma \sim \tau \quad \Leftrightarrow \sigma(y) = \tau(y).$$

The characteristic polynomial of y is

$$\chi_y(X) = \prod_{\sigma \in \Lambda} (X - \sigma(y))$$

and we conclude that there exists a matrix $C \in \mathrm{GL}_m(E')$ such that

$$C^{-1}\mu(y)C = \mathrm{diag}(\sigma_1(y), \dots, \sigma_n(y))$$

where $\Lambda = \{\sigma_1, \dots, \sigma_n\}$. Now let y be a primitive element of E'/E. We may write every $x \in E'$ as

$$x = \sum_{i=0}^{n-1} x_i y^i \quad \text{with } x_i \in E.$$

Thus we have

$$C^{-1}\mu(x)C = \sum_{i=0}^{n-1} x_i \left(C^{-1}\mu(y)C\right)^i = \mathrm{diag}(\sigma_1(x), \dots, \sigma_n(x))$$

which shows that the matrices $\{\mu(x) |\ x \in E'\}$ are simultaneously diagonalizable to the above form. For $Y = (y_{ij})$ we compute

$$\mathrm{diag}(C, \dots, C)^{-1}\mu(Y)\,\mathrm{diag}(C, \dots, C) =$$

$$\begin{pmatrix}
\sigma_1(y_{11}) & & & \sigma_1(y_{1m}) & & \\
& \ddots & \cdots & & \ddots & \\
& & \sigma_n(y_{11}) & & & \sigma_n(y_{1m}) \\
& \vdots & & \ddots & \vdots & \\
\sigma_1(y_{m1}) & & & \sigma_1(y_{mm}) & & \\
& \ddots & \cdots & & \ddots & \\
& & \sigma_n(y_{m1}) & & & \sigma_n(y_{mm})
\end{pmatrix}$$

and obtain the similar matrix

$$\begin{pmatrix} \sigma_1(Y) & & 0 \\ & \ddots & \\ 0 & & \sigma_n(Y) \end{pmatrix}$$

after applying the same number of permutations to the rows and columns. This proves the assertion. □

1.3 A descent criterion

Let K be a field of characteristic 0 and let $\overline{K}$ be an algebraic closure of K. Let Γ denote the Galois group of $\overline{K}$ over K. Consider a differential equation defined by a matrix with entries in the function field $\overline{K}(t)$. The classical Picard-Vessiot theory yields a Picard-Vessiot ring $\tilde{R}$ over $\overline{K}(t)$ and we know that $\tilde{R}$ descents to a pseudo Picard-Vessiot ring over $K(t)$ if and only if there exists a continuous Γ-action on $\tilde{R}$. We want to give a practical criterion for the existence of such an action.

Theorem 1.25. *Let $A \in M_n(\overline{K}(t))$. Suppose that for every $\sigma \in \Gamma$ there exists a matrix $C \in \mathrm{GL}_n(\overline{K}(t))$ such that*

$$C(\partial - A)C^{-1} = \partial - \sigma(A).$$

(We note that the set $\{\sigma(A)|\ \sigma \in \Gamma\}$ is finite due to the continouity of the Γ-action.) Let $\tilde{R}/\overline{K}(t)$ be a Picard-Vessiot ring for A. Then there exists a K-structure on $\tilde{R}$.

Proof. We show that there exists a continuous Γ-action on $\tilde{R}$. Consider a regular point $z = t - a$ of $\partial - A$ with $a \in K$. By direct computation, we obtain a fundamental solution matrix Y with coordinates in the ring $\overline{K}[[z]]$ of formal power series. It is clear that $\overline{K}(t)[Y_{ij}, \det(Y)^{-1}]$ is a Picard-Vessiot ring, thus, without loss of generality, we may suppose that $\tilde{R} = \overline{K}(t)[Y_{ij}, \det(Y)^{-1}]$. Because $a \in K$, Γ acts via differential automorphisms on $\overline{K}[[z]]$ by action on the coefficients of the series. We remark that if the order of Γ is infinite, this action is not continuous, in general.
But as we prove now, $\tilde{R}$ is stable under Γ and the restriction of the Γ-action to $\tilde{R}$ is continuous. To this end, let $\sigma \in \Gamma$ and consider the matrix $\sigma(Y)$ with coordinates in $\overline{K}[[z]]$. It satisfies

$$\partial(\sigma(Y)) = \sigma(\partial(Y)) = \sigma(A)\sigma(Y)$$

which shows that $\sigma(Y)$ is a fundamental matrix for the equation defined by $\sigma(A)$. By assumption there exists a matrix $C \in \mathrm{GL}_n(\overline{K}(t))$ such that

$$C(\partial - A)C^{-1} = \partial - \sigma(A)$$

showing that the matrix CY is a fundamental matrix for $\partial - \sigma(A)$. But then

$$\partial(\sigma(Y)^{-1}CY) = -\sigma(Y)^{-1}\sigma(A)CY + \sigma(Y)^{-1}\sigma(A)CY = 0,$$

thus, there exists a matrix $D \in \mathrm{GL}_n(\overline{K})$ such that $CYD = \sigma(Y)$. It follows that $\sigma(Y)$ has coordinates in $\tilde{R}$ and with

$$\sigma(\det(Y)) = \det(\sigma(Y)) = \det(C)\det(Y)\det(D),$$

it is clear that we have $\sigma(\det(Y)^{-1}) \in \tilde{R}$, too. Therefore, the ring $\tilde{R}$ is stable under Γ. It remains to prove that the Γ-action on $\tilde{R}$ is continuous. Given $y \in \tilde{R}$, we have to show that y is fixed by a normal subgroup of finite index in Γ. To this end, we prove that the coefficients $y_i \in \overline{K}$ in the power series expansion

$$y = \sum_{i=0}^{\infty} y_i z^i$$

of y lie in a finite Galois extension of $K(t)$.
Since y is an element of the Picard-Vessiot ring $\tilde{R}$, it is differentially finite and thus a solution of a differential operator

$$\sum_{i=0}^{r} a_i \partial^i(y) = 0 \tag{1.1}$$

with $a_i \in \overline{K}(t)$. We have

$$\begin{array}{lclllllllll}
y & = & y_0 & + & y_1 & z & + & y_2 & z^2 & + & y_3 \; z^3 \; + \ldots \\
\partial(y) & = & y_1 & + & 2y_2 & z & + & 3y_3 & z^2 & + & \ldots \\
\partial^2(y) & = & 2y_2 & + & 3\cdot 2y_3 & z & + & \ldots & & & \\
 & & & & & \vdots & & & & & \\
\partial^r(y) & = & \multicolumn{9}{l}{\sum_{i=0}^{\infty}(i+r)(i+r-1)\ldots(i+1)y_{i+r}z^i}
\end{array}$$

and together with (1.1) we obtain a formula

$$\sum_{j=0}^{r} a_j q_{kj} y_{k-r+j} = 0, \;\; q_{kj} \in \mathbb{Q}$$

for every $k \geq r$. Applying this formula recursively, we see that the coefficients y_k for $k \geq r$ are $\mathbb{Q}(a_0, \ldots, a_r)$-linear combinations of the coefficients $y_0, \ldots, y_{r-1}$. But this implies that all coefficients y_i lie in the field $K(t)(a_0, \ldots, a_r, y_0, \ldots, y_{r-1})$ which is a finite field extension of $K(t)$. Therefore, this extension is invariant under a normal subgroup Δ of finite index in Γ. By construction this group Δ leaves y invariant. $\square$

We remark that in particular the conditions of the theorem hold if the matrix A has entries in the field $K(t)$. Therefore, we immediately see that for $A \in \mathrm{M}_n(K(t)$ a pseudo Picard-Vessiot ring over $K(t)$ exists. In the most cases it is not true that this pseudo Picard-Vessiot ring is unique up to isomorphism in $\mathsf{DRing}_{K(t)}$ as may be seen in the following chapters.
However, if we use power series to obtain a fundamental matrix Y as in the above theorem, then we may normalize Y. Namely, we can require $Y(0) \in \mathrm{GL}_n(K)$. If $\tilde{Y}$ is another fundamental matrix satisfying this condition then we have $\tilde{Y} = YC$ for some $C \in \mathrm{GL}_n(\overline{K})$ but due to the normalization we have in fact $C \in \mathrm{GL}_n(K)$. Therefore, every normalized fundamental matrix generates the same pseudo Picard-Vessiot extension.

1.4 Descent of the differential Galois group scheme

Throughout this section we use the following notation. Let K be a field of characteristic 0 and L/K a Galois extension with group Γ. Suppose that K is the field of constants of a differential field F and let $\tilde{F}$ denote the differential field F_L with L as its field of constants.

Theorem 1.26. *Let $\tilde{R}$ be a pseudo Picard-Vessiot ring over $\tilde{F}$ and let R be a K-structure on $\tilde{R}$. Then there exists an affine algebraic group scheme $\mathcal{G}$ over K such that $\mathcal{G}(K) \cong \mathrm{Aut}_{\mathsf{DRing}_F}(R)$ and $\mathcal{G}(L) \cong \mathrm{Aut}_{\mathsf{DRing}_{\tilde{F}}}(\tilde{R})$.*

Proof. The proof is completely analogous to the proof of Theorem 1.27 [PS03]. We construct a covariant functor $\mathcal{G} : \mathsf{Alg}_K \longrightarrow \mathsf{Groups}$ from the category of commutative K-algebras into the category of groups and show that it is representable.

Construction of $\mathcal{G}$.
For $B \in \mathsf{Alg}_K$ we provide $R_B := R \otimes_K B$ with the derivation defined by

$$\partial(r \otimes b) = \partial(r) \otimes b$$

where $r \in R$ and $b \in B$. In the same way, we define a derivation on $F_B := F \otimes_K B$. We define the functor $\mathcal{G}$ as follows. A K-Algebra B is mapped to the group of F_B-linear differential automorphisms of R_B. A K-linear map $f : B \longrightarrow B'$ is mapped to the base change map $\varphi \mapsto \varphi \otimes_B B'$.
By construction we have

$$\mathcal{G}(K) = \mathrm{Aut}_{\mathsf{DRing}_F}(R)$$

and

$$\mathcal{G}(L) = \mathrm{Aut}_{\mathsf{DRing}_{\tilde{F}}}(\tilde{R}).$$

Representability of $\mathcal{G}$.
Since $R \in \mathsf{PPV}_F$ we know that it is generated by a fundamental matrix $Y \in \mathrm{GL}_n(R)$ of an equation

$$\partial(Y) = AY$$

where $A \in M_n(F)$. Let $F[\mathrm{GL}_n] = F[X_{ij}, \det(X)^{-1}]$ denote the coordinate ring of the group scheme GL_n over F. We define a derivation on $F[\mathrm{GL}_n]$ by $\partial(X) = AX$. The kernel of the surjective differential homomorphism

$$F[\mathrm{GL}_n] \longrightarrow R, \ X \mapsto Y$$

is a maximal differential ideal Q because R is simple. This proves that we have

$$R \cong F[\mathrm{GL}_n]/Q$$

as differential rings.
The functor $- \otimes_K B$ is exact for every K-algebra B and so

$$R_B \cong F_B[\mathrm{GL}_n]/Q_B.$$

Every F_B-linear automorphism φ of R_B is therefore given by an automorphism of $F_B[\mathrm{GL}_n]$

$$X \mapsto \varphi(X) = XC, \ C \in \mathrm{GL}_n(F_B[\mathrm{GL}_n])$$

which stabilizes the ideal Q_B.
Moreover,

$$\partial(\varphi(X)) = \partial(X)C + X\partial(C) = \tilde{A}XC + X\partial(C)$$

and

$$\varphi(\partial(X)) = \varphi(\tilde{A}X) = \tilde{A}XC,$$

which shows that if φ is a differential automorphism then $\partial(C) = 0$, i.e., $C \in \mathrm{GL}_n(B)$. This yields an injective group homomorphism

$$\mathcal{G}(B) \longrightarrow \mathrm{GL}_n(B)$$

which allows us to identify $\mathcal{G}(B)$ with a subgroup of $\mathrm{GL}_n(B)$. Now we take for B the K-algebra $K[\mathrm{GL}_n] = K[Z_{ij}, \det(Z)^{-1}]$ for which we have the interpretation of $Z \in \mathrm{GL}_n(B)$ as a "universal invertible matrix". The idea is to define relations in the entries of this universal matrix in order to obtain a differential automorphism of R_B via $X \mapsto XZ$. For $x \in F_B[\mathrm{GL}_n]$ let x^Z denote the image of x under the F_B-algebra homomorphism induced by $X \mapsto XZ$.
Note that the ideal Q defined above is finitely generated because the ring $F[\mathrm{GL}_n]$ is noetherian. We fix a set of generators $q_1, \dots, q_r$ of Q. Of course, Q_B is generated by the same elements.
We choose a K-basis (e_i) of R and write

$$q_j^Z \bmod Q_B = \sum_i e_i \otimes \lambda_{ij},$$

where $\lambda_{ij} \in B$. Let $P \subset B$ be the ideal generated by $\{\lambda_{ij}\}$. We claim that the functor $\mathcal{G}$ is represented by the K-algebra B/P which we denote by U.
Indeed, let $B' \in \mathsf{Alg}_K$ and $C \in \mathrm{GL}_n(B')$. The K-algebra homomorphism

$$\phi : B \longrightarrow B', \ Z \mapsto C$$

factors over U if and only if the λ_{ij} lie in the kernel of ϕ. The latter is the case if and only if the ideal $Q_{B'}$ is stable under $\cdot^{\,C}$ or, in other words, $C \in \mathcal{G}(B')$. Therefore we have

$$\mathrm{Hom}_K(U, B') \cong \mathcal{G}(B')$$

for every K-algebra B' and we check that this defines a natural transformation.
Thus, the functor $\mathcal{G}$ corresponds to the affine group scheme $\mathrm{Spec}(U)$ and we just write $\mathcal{G} = \mathrm{Spec}(U)$. □

Given a pseudo Picard-Vessiot ring R over F with corresponding pseudo Picard-Vessiot field E, remark that the groups $\mathrm{Aut}_{\mathsf{DRing}_F}(R)$ and $\mathrm{Aut}_{\mathsf{DRing}_F}(E)$ are naturally isomorphic.

Definition 1.27. *The affine group scheme in Theorem 1.26 is called the* **Galois group scheme of** R **over** F *or the* **Galois group scheme of** E **over** F *where* $E = \mathrm{Quot}(R)$. *It is denoted by* $\mathrm{Gal}_D(R/F)$ *respectively* $\mathrm{Gal}_D(E/F)$. *As the proof of Theorem 1.26 shows, we used a generating fundamental matrix* Y *of* R *in order to represent* $\mathcal{G}$. *If we want to point this out then we say that* $\mathcal{G}$ *is the* **Galois group scheme of** R **over** F **represented via** Y. *Different generating fundamental matrices lead to Galois group schemes that are isomorphic over* K.

In general, different L/K-forms of a pseudo Picard-Vessiot ring R over F have different L/K-forms of the Galois group scheme of R as Galois group schemes. We give a detailed description of this phenomenon in section 1.6.

Corollary 1.28. *If L is algebraically closed then $\mathcal{G}(L)$ is a linear algebraic group over L which is defined over K with $\mathcal{G}(K)$ as its K-rational points.*

Proof. This is clear because affine group schemes are reduced ([Wat97, 11.4, Theorem]) and the ring U in the proof of Theorem 1.26 is finitely generated. □

1.5 L/K-forms of pseudo Picard-Vessiot rings

We keep the notation of the previous sections. So far we were concerned about the existence of K-structures on a pseudo Picard-Vessiot ring $\tilde{R}$ over $\tilde{F}$. The natural question that arises after solving that problem is how many different K-structures occur on $\tilde{R}$. More precisely, we say that two K-structures are equivalent if they are isomorphic over F. Then, we want to classify equivalence classes of L/K-forms of a given pseudo Picard-Vessiot ring R over F.
The tool that we employ in order to solve this problem is Galois cohomology. We refer to [Ser97] for an exposition of this theory.

Theorem 1.29. *Let $R \in \mathsf{PPV}_F$ with Galois group scheme $\mathcal{G}$ over K. Then there is a bijection of sets*

$$\Phi : \mathrm{H}^1(\Gamma, \mathcal{G}(L)) \longrightarrow \{L/K\text{-forms of } R\}/\cong_F$$

such that $\Phi(1) = R$.

Proof. Consider $\mathcal{G}(L)$ as the group of F_L-linear differential automorphisms of R_L. Let ρ denote the natural Γ-action on R_L. By Lemma 1.7, R is the ring of invariants under this action. The group Γ acts on $\mathcal{G}(L)$ via

$$\sigma(g) = \rho(\sigma) \circ g \circ \rho(\sigma)^{-1}$$

for $\sigma \in \Gamma$ and $g \in \mathcal{G}(L)$. A 1-cocycle χ of Γ in $\mathcal{G}(L)$ satisfies

$$\chi(\sigma\tau) = \chi(\sigma) \circ \sigma(\chi(\tau)).$$

The idea is, that all L/K-forms of R can be obtained by twisting ρ and then taking invariants under the twisted action.
Let $[\chi] \in \mathrm{H}^1(\Gamma, \mathcal{G}(L))$ represented by χ. We define the twisted Γ-action

$$\rho_\chi(\sigma) = \chi(\sigma) \circ \rho(\sigma)$$

on R_L. By Theorem 1.23, the ring of invariants R_χ under this action is a pseudo Picard-Vessiot ring over F. Let χ' be a cocycle which is cohomologous to χ, i.e., there exists an element $g \in \mathcal{G}(L)$ such that

$$\chi(\sigma) = g^{-1} \circ \chi'(\sigma) \circ \sigma(g)$$

for all $\sigma \in \Gamma$. Since

$$\begin{aligned}\rho_{\chi'}(\sigma) \circ g &= \chi'(\sigma) \circ \rho(\sigma) \circ g \\ &= \chi'(\sigma) \circ \sigma(g) \circ \rho(\sigma) \\ &= g \circ \chi(\sigma) \circ \rho(\sigma) = g \circ \rho_{\chi}(\sigma)\end{aligned}$$

for every $\sigma \in \Gamma$, g induces an F-linear isomorphism of R_χ onto $R_{\chi'}$. Therefore we obtain a well defined map by setting

$$\Phi([\chi]) = [R_\chi]_{\cong_F}.$$

On the other hand let R' be an L/K-form of R. The group Γ acts on the set $\mathrm{Hom}_{\mathsf{DRing}_{\tilde{F}}}(R_L, R'_L)$ via

$$\sigma(\psi) = \rho_{R'}(\sigma) \circ \psi \circ \rho_R(\sigma)^{-1}$$

where $\sigma \in \Gamma$ and $\psi \in \mathrm{Hom}_{\mathsf{DRing}_{\tilde{F}}}(R_L, R'_L)$ and the Γ-actions are the natural ones. Since by definition there exists an isomorphism $\psi : R_L \longrightarrow R'_L$, we can define the map

$$\chi(\sigma) = \psi^{-1} \circ \sigma(\psi)$$

and the calculation

$$\begin{aligned}\chi(\sigma\tau) &= \psi^{-1} \circ (\sigma\tau)(\psi) \\ &= \psi^{-1} \circ \rho_{R'}(\sigma\tau) \circ \psi \circ \rho_R(\sigma\tau)^{-1} \\ &= (\psi^{-1} \circ \rho_{R'}(\sigma) \circ \psi \circ \rho_R(\sigma)^{-1}) \circ \rho_R(\sigma) \circ (\psi^{-1} \circ \rho_{R'}(\tau) \circ \psi \circ \rho_R(\tau)^{-1}) \circ \rho_R(\sigma)^{-1} \\ &= \chi(\sigma) \circ \sigma(\chi(\tau))\end{aligned}$$

proves that χ is a cocycle of Γ in $\mathcal{G}(L)$. If $\psi' : R_L \longrightarrow R'_L$ is another isomorphism then

$$(\psi^{-1} \circ \psi')^{-1} \circ \chi(\sigma) \circ \sigma(\psi^{-1} \circ \psi') = \chi'(\sigma)$$

where $\chi'(\sigma) = \psi'^{-1} \circ \sigma(\psi')$ and therefore χ' is cohomologous to χ. An easy computation shows that if R' and R'' is an L/K-form of R with $R' \cong_F R''$ then the above construction leads to cohomologous cocycles. This yields the well defined map

$$[R']_{\cong_F} \mapsto [\chi_{R'}]$$

which can be easily seen to be the inverse map of Φ.

□

In the proof of the above theorem, we let Γ act on $\mathcal{G}(L) = \mathrm{Aut}_{\mathsf{DRing}_{\tilde{F}}}(\tilde{R})$ via

$$\sigma(\phi) = \rho(\sigma) \circ g \circ \rho(\sigma)^{-1} \tag{1.2}$$

where $\sigma \in \Gamma$ and $g \in \mathcal{G}(L)$. The group $\mathcal{G}(L)$ may be identified with the group of matrices in $\mathrm{GL}_n(L)$ such that

$$Y \mapsto YC \tag{1.3}$$

defines a $\tilde{F}$-linear differential automorphism of $\tilde{R}$, where $Y \in \mathrm{GL}_n(R)$ is a generating fundamental matrix of R. We used this fact to prove that $\mathcal{G}$ is represented by a K-algebra U.

Of course, the action (1.2) induces an action on the L-valued points $\mathcal{G}(L) \subset \mathrm{GL}_n(L)$ of the representing scheme $\mathrm{Spec}(U)$ which is just given by applying σ to each entry of $C \in \mathrm{GL}_n(L)$ as one can easily compute from formula (1.3).

1.6 L/K-forms of the Galois group scheme

Let $\tilde{R}$ be a pseudo Picard-Vessiot ring over $\tilde{F}$ with a K-structure R corresponding to the Γ-action ρ on $\tilde{R}$. In Theorem 1.26 we defined the Galois group scheme $\mathcal{G}$ of R over F as a functor which turned out to be representable by a K-algebra U, i.e., $\mathcal{G} = \mathrm{Spec}(U)$. We also say that U is the coordinate ring of $\mathcal{G}$. The fact that U represents a group functor implies that we have a Hopf algebra structure on U. Morphisms of affine schemes over K correspond to K-algebra maps of the respective coordinate rings. If we consider affine group schemes, we only permit K-algebra maps that additionally respect the given Hopf algebra structure on the coordinate rings. This yields the category GSch_K of affine group schemes over K which is anti-equivalent to the category of Hopf algebras over K
Again we define a base change functor sending $\mathcal{G} = \mathrm{Spec}(U)$ to $\mathcal{G}_L = \mathrm{Spec}(U_L)$ where the Hopf algebra structure on U_L is canonical, thus $\mathcal{G}_L \in \mathsf{GSch}_L$. We call $\mathcal{G}$ a K-structure on $\mathcal{G}_L$ and it is clear that every K-structure on $\mathcal{G}_L$ corresponds to a continuous Γ-action on $\mathcal{G}_L$, i.e., a group homomorphism

$$\rho : \Gamma \longrightarrow \mathrm{Aut}_{\mathsf{GSch}_L}(\mathcal{G}_L).$$

Given group schemes $\mathcal{G}$, $\mathcal{G}'$ over K, we call $\mathcal{G}'$ an L/K-form of $\mathcal{G}$ if $\mathcal{G}_L \cong \mathcal{G}'_L$. In analogy to Theorem 1.29, we have the following theorem.

Theorem 1.30. *Let $\mathcal{G}$ be an affine group scheme over K. Then there is a bijection between the set $\mathrm{H}^1(\Gamma, \mathrm{Aut}_{\mathsf{GSch}_L}(\mathcal{G}_L))$ and the K-isomorphism classes of L/K-forms $\mathcal{G}$ under which the trivial class is mapped to $\mathcal{G}$.*

Proof. The proof is exactly the same as the proof of Theorem 1.29 replacing differential structure by Hopf algebra structure. We just give a sketch of the proof. Of course, the problem is equivalent to classifying L/K-forms of Hopf algebras over K. Let U be the Hopf algebra such that $\mathcal{G} = \mathrm{Spec}(U)$. The natural Γ-action on U_L is denoted by ρ_U. Given a 1-cocycle χ we twist ρ_U to obtain an L/K-form of U by taking invariants. We remark that the ring of invariants of this twisted action inherits the Hopf algebra structure of U_L since the action ρ_U and the cocycles respect this structure. This defines a map

$$\Phi : \mathrm{H}^1(\Gamma, \mathrm{Aut}_{\mathsf{GSch}_L}(\mathcal{G}_L)) \longrightarrow \{L/K\text{-forms of } U\} / \cong_K .$$

On the other hand let U' be a K-algebra such that $U_L \cong U'_L$. With the natural Γ-action $\rho_{U'}$ on U'_L we define

$$\sigma(\psi) = \rho_{U'}(\sigma) \circ \psi \circ \rho_U(\sigma)^{-1},$$

where $\sigma \in \Gamma$ and $\psi : U_L \longrightarrow U'_L$ is a Hopf algebra homomorphism. If we let ψ be the isomorphism $U_L \longrightarrow U'_L$ then the map

$$\chi_{R'}(\sigma) = \psi^{-1} \circ \sigma(\psi)$$

defines a 1-cocycle and induces a map which is inverse to Φ. The details can be copied word by word from the proof of Theorem 1.29. □

Let $\tilde{R}$ be a pseudo Picard-Vessiot ring over $\tilde{F}$ with field of constants L and let R be a K-structure on $\tilde{R}$. Applying the base change functor to the Galois group scheme $\mathcal{G}$ of

R over F yields the Galois group scheme $\mathcal{G}_L$ of $\tilde{R}$ over $\tilde{F}$. Let R' be an L/K-form of R. The Galois group scheme $\mathcal{G}'$ of R' is an L/K-form of $\mathcal{G}$. This is clear because after base extension to L the group functors $\mathcal{G}'_L$ and $\mathcal{G}_L$ are isomorphic as can be directly seen from the definition. More precisely,

$$\mathcal{G}(B) = \mathrm{Aut}_{\mathsf{DRing}_{F_B}}(R \otimes_K B) \quad \text{and} \quad \mathcal{G}'(B) = \mathrm{Aut}_{\mathsf{DRing}_{F_B}}(R' \otimes_K B)$$

for any K-algebra B and therefore $\mathcal{G}'_L \cong \mathcal{G}_L$ since $R_L \cong R'_L$.
This leads to the question which L/K-forms of $\mathcal{G}$ are actually realized as Galois group schemes of L/K-forms of R. Theorem 1.30 identifies the set of (equivalence classes of) L/K-forms of $\mathcal{G}$ with the set $\mathrm{H}^1(\Gamma, \mathrm{Aut}_{\mathsf{GSch}_L}(\mathcal{G}_L))$. We want to characterize those cohomology classes which correspond to realized forms of $\mathcal{G}$.
To this end, let $U = K[Z_{ij}, \det(Z)^{-1}]$ be the Hopf algebra of $\mathcal{G}$. We define the group homomorphism

$$\mathrm{Inn} : \mathcal{G}(L) \longrightarrow \mathrm{Aut}_{\mathsf{GSch}_L}(\mathcal{G}_L),$$

by letting $\mathrm{Inn}(C)$ be the automorphism induced by

$$U_L \longrightarrow U_L, \quad Z \mapsto C^{-1}ZC,$$

where we consider $\mathcal{G}(L)$ as a subgroup of $\mathrm{GL}_n(L)$. Thus, the image of Inn are the inner automorphisms of $\mathcal{G}_L$. The homomorphism Inn is Γ-equivariant with respect to the actions on $\mathcal{G}(L)$ and $\mathrm{Aut}_{\mathsf{GSch}_L}(\mathcal{G}_L)$ that we used above (in the classification results Theorem 1.29 and Theorem 1.30, respectively). Therefore Inn induces a map

$$\mathrm{Inn}_* : \mathrm{H}^1(\Gamma, \mathcal{G}(L)) \longrightarrow \mathrm{H}^1(\Gamma, \mathrm{Aut}(\mathcal{G}_L))$$

on the cohomology sets. An L/K-form of $\mathcal{G}$ that corresponds to a class in $\mathrm{Inn}_*(\mathrm{H}^1(\Gamma, \mathcal{G}(L)))$ is called an **inner form of $\mathcal{G}$**.

Theorem 1.31. *Exactly the inner forms of $\mathcal{G}$ occur as Galois group schemes of L/K-forms of R.*

Proof. Let $U = K[Z_{ij}, \det(Z)^{-1}]$ be the Hopf algebra of $\mathcal{G}$. As remarked in Definition 1.27, the algebra U does not only depend on R but also on the generating fundamental matrix Y of R. We thus say that U is the Hopf algebra for (R, Y). Let R' be an L/K-form of R and let ρ_R denote the natural Γ-action on R_L. According to Theorem 1.29 and its proof we obtain R' by twisting ρ_R with a cocycle χ in $\mathcal{G}(L)$ and taking invariants under the twisted action ρ_χ. We identify $\mathcal{G}(L)$ with a subgroup of $\mathrm{GL}_n(L)$. By Hilbert's Theorem 90 the cocyle χ splits in the group $\mathrm{GL}_n(L)$, i.e., there exists a matrix $C \in \mathrm{GL}_n(L)$ such that

$$\chi(\sigma) = C^{-1}\sigma(C)$$

for all $\sigma \in \Gamma$. We define $Y' = YC^{-1}$ and compute for every $\sigma \in \Gamma$

$$\rho_\chi(\sigma)(Y') = \chi(\sigma)(\rho_R(\sigma)(YC^{-1})) = \chi(\sigma)(Y\sigma(C^{-1})) = YC^{-1}\sigma(C)\sigma(C^{-1}) = Y'.$$

We conclude that the fundamental solution matrix $Y' \in \mathrm{GL}_n(R')$ generates R'. Let $U' = K[Z'_{ij}, \det(Z')^{-1}]$ be the Hopf algebra for (R', Y'). The isomorphism

$$\gamma_C : U_L \longrightarrow U'_L, \quad Z \mapsto C^{-1}Z'C$$

of L-algebras is well defined by construction of U and U' (this is due to the equation $Y' = YC^{-1}$). Since conjugation is a group homomorphism, γ_C respects the Hopf structure. We compute the cocycle ξ of Γ in $\mathrm{Aut}_{\mathsf{GSch}_L}(\mathcal{G}_L)$ induced by γ_C (see the proof of Theorem 1.30):

$$\begin{aligned}\xi(\sigma)(Z) &= \gamma_C^{-1} \circ \sigma(\gamma_C)(Z)\\ &= \gamma_C^{-1}(\sigma(C^{-1})Z'\sigma(C))\\ &= \sigma(C^{-1})CZC^{-1}\sigma(C)\\ &= \chi(\sigma)^{-1}Z\chi(\sigma).\end{aligned}$$

Thus the class of ξ lies in the image of Inn_*. Suppose we start with another L/K-form R'' of R having the Galois group scheme $\mathcal{G}'' = \mathrm{Spec}(U'')$ which is isomorphic to $\mathcal{G}'$ over K. Then the above procedure produces a map

$$\gamma_{C'} : U_L \longrightarrow U''_L$$

which induces a cocycle in $\mathrm{Inn}_*(\mathrm{H}^1(\Gamma, \mathcal{G}(L)))$ that is cohomologous to ξ by Theorem 1.30 (and its proof). We obtain a well defined map

$$\Phi : \left\{ \begin{array}{c}\text{Galois group schemes}\\ \text{of } L/K\text{-forms of } R\end{array} \right\} / \cong_K \longrightarrow \mathrm{Inn}_*(\mathrm{H}^1(\Gamma, \mathcal{G}(L))).$$

On the other hand, let ξ be a representative of an inner cohomology class such that ξ is the conjugation with a cocycle χ of Γ in $\mathcal{G}(L)$. Then we can twist the action ρ_R with χ to obtain an L/K-form R' of R as ring of invariants of the twisted action. We map ξ to the K-isomorphism class of the Galois group scheme $\mathcal{G}'$ of R'. In general, choosing a cocyle ξ' cohomologous to ξ which is conjugation with χ' leads to an L/K-form R'' which is not K-isomorphic to R'. However, the Galois group scheme $\mathcal{G}''$ of R'' is K-isomorphic to $\mathcal{G}'$. We can check this by applying the map Φ to $\mathcal{G}''$ and $\mathcal{G}'$ using R'' and R' as realizing L/K-forms. This yields the same cohomology classes proving that $\mathcal{G}''$ and $\mathcal{G}'$ are isomorphic (Φ is clearly injective since it is just a restriction of the bijective map from Theorem 1.30).
This defines a map which is inverse to Φ. □

Let $R^1_1, \dots, R^1_{j_1}, \dots, R^s_1, \dots, R^s_{j_s}$ be representatives of the K-isomorphism classes of K-structures of the pseudo Picard-Vessiot ring $\tilde{R}$ over $\tilde{F}$. For each $1 \le i \le s$ let the K-structures R^i_* have Galois group schemes which are K-isomorphic and let $\mathcal{G}^i$ be a representative group scheme for each such collection. We illustrate the situation in a picture.

$$\begin{array}{ccccccc} & & & & \tilde{R} & & \\ \mathrm{H}^1(\Gamma, \mathcal{G}(L)) & \leftrightarrow & \underbrace{R^1_1 \cdots\cdots R^1_{j_1}} & \cdots & \underbrace{R^i_1 \cdots\cdots R^i_{j_i}} & \cdots & \underbrace{R^s_1 \cdots\cdots R^s_{j_s}} \\ \mathrm{Inn}_*(\mathrm{H}^1(\Gamma, \mathcal{G}(L)) & \leftrightarrow & \mathcal{G}^1 & \cdots & \mathcal{G}^i & \cdots & \mathcal{G}^s \end{array}$$

1.7 Picard-Vessiot extensions vs. pseudo Picard-Vessiot extensions

We choose $\overline{K}$ to be an algebraic closure of K such that $L \subset \overline{K}$. For an algebraic affine group scheme $\mathcal{G}$ over K it is clear how to define the structure of a linear algebraic group on $\mathcal{G}(\overline{K})$. In the next theorem we consider $\mathcal{G}(K)$ and $\mathcal{G}(L)$ as K-rational and L-rational points of the linear algebraic group $\mathcal{G}(\overline{K})$, respectively.

Theorem 1.32. *Let $\tilde{R}$ be a Picard-Vessiot ring over $\tilde{F}$ and let the pseudo Picard-Vessiot ring R over F be a K-structure on $\tilde{R}$. Let $\mathcal{G}$ be the Galois group scheme of R over F. Then R is a Picard-Vessiot ring if and only if $\mathcal{G}(K)$ is a dense subset of $\mathcal{G}(L)$ with respect to the Zariski topology on $\mathcal{G}(L)$ (the closed sets are the zero sets of polynomials with coefficients in L).*

Proof. Suppose R is a Picard-Vessiot ring over F. Let $E = \mathrm{Quot}(R)$ then we have $E^{\mathcal{G}(K)} = F$ by definition. Choose an algebraic closure $\overline{K}$ of K with $L \subset \overline{K}$. The action of $\mathcal{G}(K) \subset \mathcal{G}(\overline{K})$ on $E_{\overline{K}}$ is given by

$$g(e \otimes l) = g(e) \otimes l \text{ , where } g \in \mathcal{G}(K),\ e \in E \text{ and } l \in \overline{K}.$$

We choose a K-basis of $\overline{K}$ which is also an E-basis of $E_{\overline{K}}$. Then the action of $\mathcal{G}(K)$ on $E_{\overline{K}}$ is given by action on the coordinates with respect to this basis. From this we immediately see that

$$(E_{\overline{K}})^{\mathcal{G}(K)} = E^{\mathcal{G}(K)} \otimes \overline{K} = F_{\overline{K}}.$$

The Galois correspondence over algebraically closed fields of constants ([PS03, Proposition 1.34]) implies that $\mathcal{G}(K) \subset \mathcal{G}(\overline{K})$ is a dense subset. We claim $\mathcal{G}(K) \subset \mathcal{G}(L)$ is also dense. Suppose there is a nontrivial closed subset of $\mathcal{G}(L)$ which contains $\mathcal{G}(K)$. This subset is defined by nontrivial polynomial equations with coefficients in L. These equations also define a nontrivial closed subset of $\mathcal{G}(\overline{K})$ which contains $\mathcal{G}(K)$, contradiction.

Next suppose that $\mathcal{G}(K) \subset \mathcal{G}(L)$ is a dense subset. Given an L-valued point x on an L-variety X which is a $\mathcal{G}(L)$-space, the isotropy group $\mathcal{G}(L)_x$ is a closed L-subgroup of $\mathcal{G}(L)$ ([Spr98, 2.3.1,12.1.2]). Suppose there exists $f \in R\backslash F$ which is fixed by $\mathcal{G}(K)$. We construct a $\mathcal{G}(L)$-space X with $f \in X$. Then the isotropy group $\mathcal{G}(L)_f$ of f is a closed subgroup of $\mathcal{G}(L)$ which contains $\mathcal{G}(K)$. The density then implies that $\mathcal{G}(L)_f = \mathcal{G}(L)$ which is a contradiction to the assumption that $\tilde{R}$ is a Picard-Vessiot ring over $\tilde{F}$.
It remains to prove the existence of the space X. In analogy to the proof of [Spr98, 2.3.6] we can find a finite dimensional subspace X of the L-vector space R_L which is stable under $\mathcal{G}(L)$ and contains f. We give X the structure of a variety by identifying it with $\mathbb{A}^r(L)$ where r is the dimension of V. Thus V is a $\mathcal{G}(L)$-space with the above properties.

□

Proposition 1.33. *Let K be a field of characteristic 0 and let L be a Galois extension of K. Let G be a connected linear algebraic K-group. Then $G(K) \subset G(L)$ is a dense subset.*

Proof. By [Spr98, 13.3.10], $G(K) \subset G(\overline{K})$ is a dense subset. But then the same argument as in the proof of the first part of Theorem 1.32 implies that $G(K) \subset G(L)$ is also dense. □

We remark that, together with Theorem 1.32, this proposition implies that pseudo Picard-Vessiot rings with connected Galois group scheme are automatically Picard-Vessiot rings.

1.8 The Torsor theorem

The classical analog of the following theorem is proved in [PS03, Theorem 1.28]. We give an alternative proof which moreover generalizes it to our situation.

Theorem 1.34 (Torsor theorem). *Let R be a Picard-Vessiot ring over F with Galois group scheme $\mathcal{G}$. Then $\mathcal{X} = \operatorname{Spec}(R)$ is a $\mathcal{G}_F$-torsor over F.*

Proof. Let $\mathcal{G} = \operatorname{Spec}(U)$ with the Hopf algebra $U = K[Z_{ij}, \det(Z)^{-1}]$ and let the Picard-Vessiot ring R be generated by the fundamental matrix $Y \in \mathrm{GL}_n(R)$ for the equation defined by some $A \in \mathrm{M}_n(F)$. Suppose that $\mathcal{G}$ is represented via Y then by construction of $\mathcal{G}$ (Theorem 1.26) the F-algebra homomorphism

$$\gamma : R \longrightarrow R \otimes_F U_F, \quad Y_{ij} \mapsto \sum_{k=1}^{n} Y_{ik} \otimes Z_{kj} =: X_{ij}$$

defines a $\mathcal{G}_F$-action on $\mathcal{X}$. Thus $\mathcal{X}$ is a $\mathcal{G}_F$-scheme and it remains to prove that the natural map $\varphi := \imath_R \otimes \gamma$ which makes the diagram

$$\begin{tikzcd}
& & R \otimes_F U_F \\
R \arrow[rru, bend left, "\imath_R"] \arrow[r] & R \otimes_F R \arrow[ru, "\varphi"] & \\
F \arrow[u] \arrow[r] & R \arrow[u] \arrow[ruu, bend right, "\gamma"'] &
\end{tikzcd} \tag{1.4}$$

commutative is an isomorphism.

From (1.4) we see that φ is R-linear with respect to the first component of the tensor products. Because Y is invertible in R, the computation

$$\varphi(\sum_{k=1}^{n}(Y^{-1})_{ik} \otimes Y_{kj}) = \sum_{k=1}^{n}(Y^{-1})_{ik}\varphi(Y_{kj}) = \sum_{k=1}^{n}(Y^{-1})_{ik} \sum_{s=1}^{n} Y_{ks} \otimes Z_{sj} = Z_{ij}$$

shows that φ is surjective.

The surjectivity of φ corresponds to the fact that the $\mathcal{G}_F$-action on $\mathcal{X}$ is faithful. We next prove the injectivity which means that the action is simply transitive. Naively, one could try to just define the inverse map via

$$Z_{ij} \mapsto \sum_{k=1}^{n}(Y^{-1})_{ik} \otimes Y_{kj}$$

but it is not clear if this map is well defined. Therefore we proceed by showing that the kernel of φ is trivial.

Let $P = \mathrm{Kern}(\varphi)$ and let $E = \mathrm{Quot}(R)$ be the Picard-Vessiot field corresponding to R. We obtain the exact sequence of R-modules

$$0 \longrightarrow P \longrightarrow R \otimes_F R \xrightarrow{\varphi} R \otimes_F U_F \longrightarrow 0$$

which remains exact after localizing

$$0 \longrightarrow E \otimes_R P \longrightarrow E \otimes_F R \xrightarrow{\varphi} E \otimes_F U_F \longrightarrow 0$$

since E is a flat R-module ([Eis95, Proposition 2.5]). Since R is a domain, it suffices to show that $P_E = E \otimes_R P = 0$.
We let the group $\mathcal{G}(K)$ $(= \mathrm{Aut}_{\mathsf{DRing}_F}(E))$ act on the first component of $E \otimes_F R$. Furthermore, we obtain a $\mathcal{G}(K)$-action on $E \otimes_F U_F$ by setting

$$g(e \otimes u) = g(e) \otimes g(u) \quad , g \in \mathcal{G}(K),\, e \in E,\, u \in U_F,$$

where the action on U_F is given by the F-algebra homomorphism defined by $Z \mapsto g^{-1}Z$ (this map is well defined since $g \in \mathcal{G}(K)$).
The map φ is equivariant with respect to these actions which implies that P_E is a $\mathcal{G}(K)$-stable ideal of $E \otimes_F R$. We apply Lemma 1.17 to conclude that P_E corresponds to the ideal $P' := P_E \cap R \subset R$. Therefore it suffices to prove that P' is trivial.
We have $P' = \mathrm{Kern}(\varphi_{|R})$ and since $\varphi_{|R} = \gamma$ we obtain the exact sequence

$$0 \longrightarrow P' \longrightarrow R \xrightarrow{\gamma} R \otimes_F U_F.$$

Because $R \otimes_F U_F \cong R \otimes_K U$, we may extend the derivation on R to $R \otimes_F U_F$ by setting $\partial(u) = 0$ for $u \in U$. With respect to this derivation we compute $\partial(X) = AX$ which shows that γ is a differential homomorphism. Finally, we conclude that P' is a differential ideal and thus $P' = 0$ since R is a simple differental ring. □

Corollary 1.35. *Theorem 1.34 remains true for pseudo Picard-Vessiot rings.*

Proof. Let R be a pseudo Picard-Vessiot ring over F. By Proposition 1.18, the ring $R_{\overline{K}} = R \otimes_F (F \otimes_K \overline{K}) = R \otimes_K \overline{K}$ is a pseudo Picard-Vessiot ring over $F \otimes_K \overline{K}$. From classical Picard-Vessiot theory, we conclude that $R_{\overline{K}}$ is in fact a Picard-Vessiot ring. We have to show that the homomorphism

$$\varphi : R \otimes_F R \longrightarrow R \otimes_F U_F$$

defined in the proof of the theorem is an isomorphism. But this is clear since $\overline{K}$ is faithfully flat over K and the homomorphism

$$\varphi_{\overline{K}} : R_{\overline{K}} \otimes_{F_{\overline{K}}} R_{\overline{K}} \longrightarrow R_{\overline{K}} \otimes_{F_{\overline{K}}} U_{F_{\overline{K}}}$$

is an isomorphism by the theorem. □

Theorem 1.36. *Let F be a field and let $\mathcal{G}$ be an affine group scheme of finite type over F. Then*

$$\mathrm{H}^1(\Gamma_F, \mathcal{G}(\overline{F})) \cong \{\mathcal{G}\text{-torsors over } F\}/ \cong_F \,,$$

where Γ_F denotes the absolute Galois group of F.

Proof. [Wat97, p.143, Corollary] □

Let $\tilde{F} = \overline{K}(t)$. We use the torsor theorem to show that every Picard-Vessiot ring $\tilde{R}$ over $\tilde{F}$ with connected Galois group scheme $\tilde{\mathcal{G}}$ is effective. Suppose that $\tilde{R}$ is generated by the fundamental matrix $Y \in \mathrm{GL}_n(\tilde{R})$ for an equation defined by $A \in \mathrm{M}_n(\tilde{F})$. The effectiveness means that there is a matrix $\tilde{A} \in \mathrm{Lie}_{\tilde{F}}(\tilde{\mathcal{G}})$ which is equivalent to A over $\tilde{F}$. The condition $\mathrm{cd}(\tilde{F}) \leq 1$ implies that the $\tilde{\mathcal{G}}$-torsor $\tilde{\mathcal{X}} = \mathrm{Spec}(\tilde{R})$ is trivial over $\tilde{F}$ by [PS03, Theorem A.53]. This is equivalent to the existence of an $\tilde{F}$-valued point $C \in \tilde{\mathcal{X}}(\tilde{F})$. Since $Y \in \tilde{\mathcal{X}}(\tilde{R})$, this means that we find $X \in \tilde{\mathcal{G}}(\tilde{R})$ such that $CX = Y$. Next we compute

$$\partial(X) = (C^{-1}AC + C^{-1}\partial(C))X$$

and by straightforward computation one can show that the matrix $\tilde{A} = C^{-1}AC + C^{-1}\partial(C)$ (which is the logarithmic derivative of X) is an element of $\mathrm{Lie}_{\tilde{F}}(\tilde{\mathcal{G}})$.
If we work over the differential field $F = K(t)$ with non algebraically closed field of constants K, this proof does not work anymore. This is because the condition $\mathrm{cd}(F) \leq 1$ is not satisfied. Therefore the torsor $\mathcal{X} = \mathrm{Spec}(R)$ is not necessarily trivial. An example for this is given in Example 2 in Section 1.10. We say that a pseudo Picard-Vessiot ring R over F with Galois group scheme $\mathcal{G}$ is K-effective if it is induced by a matrix $A \in \mathrm{Lie}_F(\mathcal{G})$. As may be seen in Example 2 in Section 1.10 a pseudo Picard-Vessiot ring may be K-effective even if the corresponding torsor is not trivial.

1.9 The Galois correspondence

The next theorem generalizes the Galois correspondence from classical Picard-Vessiot theory. It is proved by descent from the classical result. The connected component of a group scheme is constructed in [Wat97, 6.7 Theorem].

Theorem 1.37. *Let K be a field of characteristic 0, let F be a differential field with field of constants K and let E be a pseudo Picard-Vessiot field over F with Galois group scheme $\mathcal{G}$. Let*

$$\mathfrak{H} = \{\mathcal{H} \leq \mathcal{G} \mid \mathcal{H} \text{ closed subgroup scheme of } \mathcal{G} \text{ over } K\}$$

as well as

$$\mathfrak{M} = \{M \mid F \subset M \subset E \text{ tower of differential fields }\}.$$

(1) The map

$$\Psi : \mathfrak{M} \longrightarrow \mathfrak{H},\ M \mapsto \mathrm{Gal}_D(E/M)$$

is an anti-isomorphism of lattices. If $\mathcal{H} \in \mathfrak{H}$, then the field $(E_{\overline{K}})^{\mathcal{H}(\overline{K})}$ is Γ-stable and we denote its field of invariants by $M(\mathcal{H})$. The inverse map of Ψ is then given by

$$\Phi : \mathfrak{H} \longrightarrow \mathfrak{M},\ \mathcal{H} \mapsto M(\mathcal{H}).$$

(2) If $\mathcal{H} \in \mathfrak{H}$ is a normal subgroup scheme of $\mathcal{G}$ then $M(\mathcal{H})$ is a pseudo Picard-Vessiot field over F with Galois group scheme $\mathcal{G}/\mathcal{H}$.

(3) Let $\mathcal{G}^o$ be the connected component of $\mathcal{G}$. Then E is a Picard-Vessiot field (!) over $M(\mathcal{G}^o)$ and $M(\mathcal{G}^o)$ is a finite field extension of F (not necessarily Galois).

Proof. (1) Let $M \in \mathfrak{M}$. Suppose that we represented the functor

$$\mathcal{G} = \mathrm{Aut}_D(E \otimes_K -/F \otimes_K -)$$

via the fundamental matrix $Y \in \mathrm{GL}_n(E)$. Then we may represent the functor $\mathcal{H} = \mathrm{Aut}_D(E \otimes_K -/M \otimes_K -)$ via the same fundamental matrix. We obtain a natural transformation of K-group functors $\mathcal{H} \longrightarrow \mathcal{G}$. By Yoneda's lemma we conclude that $\mathcal{H}$ is a K-subgroup scheme of $\mathcal{G}$. By inspection of the proof of Theorem 1.26 we see that $\mathcal{H}$ is in fact a closed subgroup scheme of $\mathcal{G}$ (it is defined by the equations defining $\mathcal{G}$ and some additional ones). Therefore the map Ψ is well defined.

Next let $\mathcal{H} \in \mathfrak{H}$. By Proposition 1.18, the field $\tilde{E} := E_{\overline{K}}$ is a pseudo Picard-Vessiot field over $\tilde{F} := F_{\overline{K}}$ which is even a Picard-Vessiot field by classical theory. Let $\tilde{M} = \tilde{E}^{\mathcal{H}(\overline{K})}$ and let $\Gamma = \mathrm{Gal}(\overline{K}/K)$. We show that the natural Γ-action on $\tilde{E}$ stabilizes $\tilde{M}$. By construction of $\mathcal{G}$ via a generating fundamental matrix $Y \in \mathrm{GL}_n(E)$ of E (proof of Theorem 1.26) , we have $\mathcal{H} \leq \mathcal{G} \leq \mathrm{GL}_n$ and thus $\mathcal{H}(\overline{K}) \leq \mathrm{GL}_n(\overline{K})$. We let Γ act on $\mathcal{H}(\overline{K})$ via action on the entries of the matrices. This action is well defined since $\mathcal{H}$ is a group scheme over K. Because we used the Γ-invariant fundamental matrix Y to represent $\mathcal{G}$ (and therefore $\mathcal{H}$), we furthermore have

$$\sigma(h)(\sigma(e)) = \sigma(h(e)), \text{ for } h \in \mathcal{H}(\overline{K}),\ \sigma \in \Gamma \text{ and } e \in \tilde{E},$$

where the action of $\mathcal{H}(\overline{K})$ on $\tilde{E}$ is via

$$\mathcal{H}(\overline{K}) \subset \mathcal{G}(\overline{K}) \cong \mathrm{Aut}_{\mathsf{DRing}_{\tilde{F}}}(\tilde{E}).$$

For $x \in \tilde{M}$ we thus compute

$$h(\sigma(x)) = \sigma(\sigma^{-1}(h)(x)) = \sigma(x), \text{ for all } h \in \mathcal{H}(\overline{K}),\ \sigma \in \Gamma,$$

which proves that $\tilde{M}$ is Γ-stable. We conclude by Theorem 1.23 that the pseudo Picard-Vessiot field E over $M(\mathcal{H}) = \tilde{M}^{\Gamma}$ is a K-structure of the Picard-Vessiot ring $\tilde{E}$ over $\tilde{M}$.

It is easy to check that the maps constructed above are inverses to one another by using the classical correspondence together with the fact that base change is faithfully flat. The other assertions immediately follow from the classical Galois correspondence and Theorem 1.23.

(2) Let $\mathcal{H}$ be a normal closed K-subgroup scheme of $\mathcal{G}$. Then $\mathcal{H}(\overline{K}) \subset \mathcal{G}(\overline{K})$ is a normal closed subgroup and the field $\tilde{M} := \tilde{E}^{\mathcal{H}(\overline{K})}$ is a Picard-Vessiot field over $\tilde{F}$ by the classical Galois correspondence. By Theorem 1.23, we obtain the Picard-Vessiot field $M(\mathcal{H})$ over F.

Next we have to check that the scheme $\mathcal{G}/\mathcal{H}$ is K-isomorphic to the Galois group scheme of M over F. By Yoneda's lemma, it suffices to show that it represents the K-group functor $\mathrm{Aut}_D(M \otimes_K -/F \otimes_K -)$. The action of Γ on the group $\mathrm{Aut}_D(E \otimes_K \overline{K}/F \otimes_K \overline{K})$ via

$$g \mapsto \sigma(g) = \rho_E(\sigma) \circ g \circ \rho_E(\sigma)^{-1}$$

is compatible with the obvious action of Γ on the $\overline{K}$-valued points of the representing group scheme $\mathcal{G}$.

We may consider an element of $(\mathcal{G}/\mathcal{H})(K)$ as an element g of $\mathcal{G}(\overline{K})$ for which $g^{-1} \circ \sigma(g) \in \mathcal{H}(\overline{K})$ (since $(\mathcal{G}/\mathcal{H})(\overline{K}) \cong \mathcal{G}(\overline{K})/\mathcal{H}(\overline{K})$). The element g corresponds to an automorphism of $E_{\overline{K}}/F_{\overline{K}}$ which restricts to an automorphism of $M_{\overline{K}}/F_{\overline{K}}$ by the classical Galois correspondence. But since $g^{-1} \circ \sigma(g) \in \mathcal{H}(\overline{K})$ induces the identity on $M_{\overline{K}}$, we conclude that g defines an element of $\mathrm{Aut}_D(M/F)$ by Theorem 1.23.
On the other hand, given an element $g \in \mathrm{Aut}_D(M/F)$ we obtain an element of $\mathrm{Aut}_D(M_{\overline{K}}/F_{\overline{K}})$ by base change which may be lifted to an automorphism of $E_{\overline{K}}/F_{\overline{K}}$ denoted by $\tilde{g}$ (this is due to the fact that $\mathrm{Aut}_D(M_{\overline{K}}/F_{\overline{K}}) \cong (\mathcal{G}/\mathcal{H})(\overline{K}) \cong \mathcal{G}(\overline{K})/\mathcal{H}(\overline{K})$ by the classical Galois correspondence). If we restrict the automorphism $\tilde{g}^{-1} \circ \sigma(\tilde{g})$ to $M_{\overline{K}}$, we obtain the identity since g was constructed by base change. Thus we conclude that

$$\tilde{g}^{-1} \circ \sigma(\tilde{g}) \in \mathcal{H}(\overline{K}),$$

i.e., $\tilde{g} \in (\mathcal{G}/\mathcal{H})(K)$.
Since these constructions commute with base change they define an isomorphism of K-group functors

$$\mathcal{G}/\mathcal{H} \cong \mathrm{Aut}_D(M \otimes_K -/F \otimes_K -)$$

which proves $\mathrm{Gal}_D(M/F) \cong \mathcal{G}/\mathcal{H}$.

(3) By the first part E is a pseudo Picard-Vessiot field over $M(\mathcal{G}^o)$. Due to Proposition 1.33 and Theorem 1.32 it is in fact a Picard-Vessiot field. By the classical Galois correspondence $\tilde{E}^{\mathcal{G}^o(\overline{K})}$ is a Galois extension of $\tilde{F}$. Thus $M(\mathcal{G}^o)/F$ is finite of the same degree (but it is not necessarily Galois).

□

1.10 Examples

We study some examples in order to get a better feeling for the descent theory. We consider the case of the Galois extension $\mathbb{C}/\mathbb{R}$ with Galois group Γ generated by the complex conjugation τ.

(1) $\mathcal{G} \cong \mathbb{G}_m$
The differential equation $\partial(y) = y$ over $\mathbb{C}(t)$ has the ring $\tilde{R} = \mathbb{C}(t)[y, y^{-1}]$ as Picard-Vessiot ring. We extend the complex conjugation to $\tilde{R}$ via $\tau(y) = y$. The ring of invariants under τ is given by $R = \mathbb{R}(t)[y, y^{-1}]$ which is a pseudo Picard-Vessiot ring by Theorem 1.23. The Galois group scheme $\mathcal{G}$ of R over $\mathbb{R}(t)$ is easily seen to be the group scheme $\mathbb{G}_m$ over $\mathbb{R}$.
By Hilbert's Theorem 90, we know that $\mathrm{H}^1(\Gamma, \mathbb{G}_m(\mathbb{C}))$ is trivial and conclude by Theorem 1.29 that R is the only $\mathbb{R}$-structure on $\tilde{R}$.
Let $\tilde{E} = \mathrm{Quot}(\tilde{R})$ with $\mathbb{R}$-structure $E = \mathrm{Quot}(R)$ which is a Picard-Vessiot field by Theorem 1.32 and Proposition 1.33 since the Galois group scheme is connected. Let μ_3 be the closed normal subgroup scheme of $\mathbb{G}_m$ defined by the equation $z^3 = 1$. The Picard-Vessiot field $\tilde{E}$ over $\tilde{E}^{\mu_3(\mathbb{C})}$ has the $\mathbb{R}$-structure E over $\mathbb{R}(t)(y^3)$. Thus, $\mathbb{R}(t)(y^3)$ is the pseudo Picard-Vessiot field which corresponds to the scheme μ_3 by Galois correspondence. Note that $E/\mathbb{R}(t)(y^3)$ is no Picard-Vessiot extension since $\mu_3(\mathbb{R}) = 1$.

(2) $\mathcal{G} \cong \mathrm{SO}_2$
Consider the differential equation $\partial(y) = iy$ over $\mathbb{C}(t)$ which induces the Picard-Vessiot ring $\tilde{R} = \mathbb{C}(t)[y, y^{-1}]$. There is an analytical interpretation of y as the function $\exp(it)$. We extend the complex conjugation to a differential automorphism of $\tilde{R}$ by letting $\tau(y) = y^{-1}$. The corresponding ring of invariants R is a pseudo Picard-Vessiot ring over $\mathbb{R}(t)$ for the equation

$$\partial(Y) = \mu(i)Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y.$$

Here we use the notation of the proof of Theorem 1.23 where we choose the $\mathbb{R}$-basis $(1, i)$ of $\mathbb{C}$. The base extension $R \otimes -$ yields the R-basis $(1 \otimes 1, 1 \otimes i)$ of $\tilde{R}$. With respect to this basis, we write

$$y = \underbrace{\frac{1}{2}(y + \tau(y))}_{\cos(t)} \otimes 1 + \underbrace{\frac{1}{2i}(y - \tau(y))}_{\sin(t)} \otimes i.$$

Still following the proof of Theorem 1.23, the matrix

$$\mu(y) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

is a generating fundamental matrix of $R = \mathbb{R}(t)[\cos(t), \sin(t)]$ with the only relation $\cos(t)^2 + \sin(t)^2 = 1$. We can also check the formula in Lemma 1.24 and compute

$$\det(\mu(y)) = \det(y)\tau(\det(y)) = yy^{-1} = 1.$$

The Galois group scheme of R over $\mathbb{R}(t)$ is the group scheme SO_2 over $\mathbb{R}$. By [Ser97, p.141], there is a bijection between $\mathrm{H}^1(\Gamma, \mathrm{SO}_2(\mathbb{C}))$ and the set of classes of quadratic forms of rank 2 which have positive discriminant. We conclude that $\mathrm{H}^1(\Gamma, \mathrm{SO}_2(\mathbb{C}))$ consists of two elements. A nontrivial cocycle χ is given by

$$\chi(\tau) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Twisting the above Γ-action ($\tau(y) = y^{-1}$) with χ we obtain the action $\tau_\chi(y) = -y^{-1}$. This action corresponds to the pseudo Picard-Vessiot ring $R' = \mathbb{R}(t)[i\cos(t), i\sin(t)]$ with the relation $(i\cos(t))^2 + (i\sin(t))^2 = -1$ and according to Theorem 1.29, the ring R' is a nontrivial $\mathbb{C}/\mathbb{R}$-form of R. A fundamental matrix for R' is given by

$$\mu_\chi(y) = \begin{pmatrix} i\sin(t) & -i\cos(t) \\ i\cos(t) & i\sin(t) \end{pmatrix}$$

with determinant $\det(y)\tau_\chi(\det(y)) = y(-y^{-1}) = -1$. The Galois group scheme of R' over $\mathbb{R}(t)$ is again SO_2. This can be seen by direct computation or by Theorem 1.30. Namely, despite of the cohomology $\mathrm{H}^1(\Gamma, \mathrm{Aut}((\mathrm{SO}_2)_\mathbb{C}) \cong Z_2$ not being trivial, there are no nontrivial inner $\mathbb{C}/\mathbb{R}$-forms of SO_2, since this group scheme is commutative. Thus the $\mathbb{C}/\mathbb{R}$-form $\mathbb{G}_m$ of SO_2 is not realized as Galois group scheme in the present example. We draw a diagram.

$$\begin{array}{ccc} & & \tilde{R} \\ \mathrm{H}^1(\Gamma, \mathrm{SO}_2(\mathbb{C})) \cong \{1,-1\} & \leftrightarrow & \underbrace{R \qquad\qquad R'} \\ \mathrm{Inn}_*(\mathrm{H}^1(\Gamma, \mathrm{SO}_2(\mathbb{C})) = 1 & \leftrightarrow & \mathrm{SO}_2 \end{array}$$

Note that both $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R')$ are $\mathcal{G}$-torsors where only $\mathrm{Spec}(R)$ is trivial. We can easily check that $\mathrm{Spec}(R')$ has no $\mathbb{R}(t)$-valued point. This is due to the fact that $\mathrm{cd}(\mathbb{R}(t)) \not\leq 1$ which has the consequence that $\mathrm{H}^1(\Gamma, \mathcal{G}(\mathbb{C}(t)))$ is in general nontrivial even for connected group schemes $\mathcal{G}$.
In addition, this example shows that the descent criterion given in Theorem 1.25 is not a necessary condition for descent.

(3) $\mathcal{G} \cong \mathrm{PSL}_2$
The differential equation $\partial(Y) = AY$ with

$$A = \begin{pmatrix} t & 1 \\ 1 & -t \end{pmatrix}$$

over $\mathbb{C}(t)$ has the group scheme SL_2 as Galois group scheme [MS96, p.17, Example 2]. The differential ring

$$\tilde{R} = \mathbb{C}(t)[x_1,x_2,x_3,x_4]/(x_1x_4 - x_2x_3 - 1) \quad \text{with} \quad \partial(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}) = A \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

is a Picard-Vessiot ring for A over $\mathbb{C}(t)$. According to Theorem 1.25, there exists an $\mathbb{R}$-structure on $\tilde{R}$ which corresponds to the Γ-action on $\tilde{R}$ given by $\tau(x_i) = x_i$ for all $1 \leq i \leq 4$. All $\mathbb{C}/\mathbb{R}$-forms of R are trivial since $\mathrm{H}^1(\Gamma, \mathrm{SL}_2(\mathbb{C})) = 1$, so this situation is not interesting for descent.
Let $\pi : \mathrm{SL}_2 \longrightarrow \mathrm{PSL}_2$ be the canonical morphism with corresponding Lie algebra homomorphism $d\pi$. The differential equation over $\mathbb{C}(t)$ defined by the matrix

$$d\pi(A) = \begin{pmatrix} 2t & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & -2t \end{pmatrix}$$

has the differential ring

$$\tilde{S} = \mathbb{C}(t)[x_ix_j \mid 1 \leq i \leq j \leq 4] \subset \tilde{R}$$

as Picard-Vessiot ring over $\mathbb{C}(t)$. It is generated by the fundamental matrix

$$\pi(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}) = \begin{pmatrix} x_1^2 & x_1x_2 & x_2^2 \\ 2x_1x_3 & x_1x_4 + x_2x_3 & 2x_2x_4 \\ x_3^2 & x_3x_4 & x_4^2 \end{pmatrix}$$

for $d\pi(A)$. By [Har02, Proposition 2.12], the diagram

$$\begin{tikzcd} \mathrm{Gal}_D(\tilde{R}/\tilde{F}) \arrow[r, "\mathrm{res}"] \arrow[d] & \mathrm{Gal}_D(\tilde{S}/\tilde{F}) \arrow[d] \\ \mathrm{SL}_2 \arrow[r, "\pi"] & \mathrm{PSL}_2 \end{tikzcd}$$

is commutative. This proves that $\tilde{S}$ is effective, i.e., $\mathrm{Gal}_D(\tilde{S}/\tilde{F}) \cong \mathrm{PSL}_2$. Clearly, the above Γ-action on $\tilde{R}$ restricts to an action on $\tilde{S}$ which defines an $\mathbb{R}$-structure S on $\tilde{S}$.

There is a bijection between $\mathrm{H}^1(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ and the classes of central simple algebras over $\mathbb{R}$ [Spr98, 12.3.5. Examples]. Representatives of these are the quaternions

$$\mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k \quad \text{with} \quad i^2 = j^2 = k^2 = -1, \quad ijk = -1$$

and $\mathbb{R}$ itself. A nontrivial cocycle χ is given by

$$\chi(\tau) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We twist our given Γ-action with this cocycle and obtain

$$\rho'(\tau)(\begin{pmatrix} x_1^2 & x_1x_2 & x_2^2 \\ 2x_1x_3 & x_1x_4 + x_2x_3 & 2x_2x_4 \\ x_3^2 & x_3x_4 & x_4^2 \end{pmatrix}) = \begin{pmatrix} x_2^2 & -x_1x_2 & x_1^2 \\ 2x_2x_4 & -x_1x_4 - x_2x_3 & 2x_1x_3 \\ x_4^2 & -x_3x_4 & x_3^2 \end{pmatrix}$$

as twisted action on $\tilde{S}$. The proof of Theorem 1.31 provides a procedure for finding a fundamental matrix generating the corresponding $\mathbb{C}/\mathbb{R}$-form S' of S. We split the cocycle χ in $\mathrm{GL}_n(\mathbb{C})$, e.g., if we define

$$C = \begin{pmatrix} -i & 0 & i \\ 0 & i & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

then

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = C^{-1}\tau(C).$$

Next the proof tells us that the matrix

$$\begin{pmatrix} x_1^2 & x_1x_2 & x_2^2 \\ 2x_1x_3 & x_1x_4 + x_2x_3 & 2x_2x_4 \\ x_3^2 & x_3x_4 & x_4^2 \end{pmatrix} C^{-1} =$$

$$= \frac{1}{2} \begin{pmatrix} i(x_1^2 - x_2^2) & -2ix_1x_2 & x_1^2 + x_2^2 \\ 2i(x_1x_3 - x_2x_4) & -2i(x_1x_4 + x_2x_3) & 2(x_1x_3 + x_2x_4) \\ i(x_3^2 - x_4^2) & -2ix_3x_4 & x_3^2 + x_4^2 \end{pmatrix}$$

is a generating fundamental matrix of S' over $\mathbb{R}(t)$. Let $\mathcal{G} = \mathrm{PSL}_2$ be the Galois group scheme of S over $\mathbb{R}$ and let $\mathcal{G}'$ be the Galois group scheme of S'. Conjugation with C yields the $\mathbb{C}$-isomorphism between $\mathcal{G}_{\mathbb{C}}$ and $\mathcal{G}'_{\mathbb{C}}$. We compute

$$C\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}C^{-1} =$$

$$= \begin{pmatrix} \frac{1}{2}(a^2+d^2-b^2-c^2) & cd-ab & \frac{i}{2}(c^2+d^2-a^2-b^2) \\ bd-ac & ad+bc & i(ac+bd) \\ \frac{i}{2}(a^2+c^2-b^2-d^2) & -i(ab+cd) & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix}.$$

From this we conclude that $\mathcal{G}'$ is represented by the Hopf algebra

$$\begin{aligned} U' = \mathbb{R}[&a^2+d^2-b^2-c^2, cd-ab, i(c^2+d^2-a^2-b^2), bd-ac, ad+bc, \\ &i(ac+bd), i(a^2+c^2-b^2-d^2), -i(ab+cd), a^2+b^2+c^2+d^2] \subset \tilde{U} \end{aligned}$$

where $\tilde{U}$ is the Hopf algebra of PSL_2 over $\mathbb{C}$. The algebra U' consists of the invariants under the Γ-action on $\tilde{U}$ which is given by twisting the natural action with the cocycle $\mathrm{Inn}_*(\chi)$. We immediately compute the resulting action to be

$$\tau : \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \mapsto \begin{pmatrix} d^2 & -cd & c^2 \\ -2bd & ad+bc & -2ac \\ b^2 & -ab & a^2 \end{pmatrix}$$

and check that the generators of U' are indeed invariants.
Again we visualize the situation in a diagram.

$$\begin{array}{lccccc} & & & \tilde{S} & \\ & & / & & \backslash \\ \mathrm{H}^1(\Gamma, \mathrm{PSL}_2(\mathbb{C})) = \{1,-1\} & \leftrightarrow & \underbrace{S} & & \underbrace{S'} \\ \mathrm{Inn}_*(\mathrm{H}^1(\Gamma, \mathrm{PSL}_2(\mathbb{C})) = \{1,-1\} & \leftrightarrow & \mathcal{G} & & \mathcal{G}' \end{array}$$

(4) $\mathcal{G} \cong \mathbb{C}^*$ (as $\mathbb{R}$-structure on $\mathbb{G}_m^2$) We study the equation defined by

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

over $\mathbb{C}(t)$. A Picard-Vessiot ring for A over $\mathbb{C}(t)$ is given by

$$\tilde{R} = \mathbb{C}(t)[x,y,z]/((x^2+y^2)z-1)$$

generated by the fundamental matrix

$$Y = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

An analytical interpretation would be

$$\begin{aligned} x &= e^t \cos(t) \\ y &= e^t \sin(t) \\ z &= e^{-2t}. \end{aligned}$$

Taking invariants under the Γ-action given by $\tau(x) = x$ and $\tau(y) = y$ yields the $\mathbb{R}$-structure

$$R = \mathbb{R}(t)[x, y, z]/((x^2 + y^2)z - 1)$$

which has the Galois group scheme defined by the Hopf algebra

$$U = \mathbb{R}[a, b, c]/((a^2 + b^2)c - 1).$$

Here the Hopf algebra structure is given by matrix operations on the universal matrix namely

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \otimes \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a \otimes a - b \otimes b & -a \otimes b - b \otimes a \\ b \otimes a + a \otimes b & -b \otimes b + a \otimes a \end{pmatrix}$$

for comultiplication and analogously for coinverse and counit. Thus we see that $\mathcal{G}$ is a $\mathbb{C}/\mathbb{R}$-form of $\mathbb{G}_m^2$ with

$$\mathcal{G}(\mathbb{R}) \cong \mathbb{C}^*.$$

By [Spr98, 12.3.4. Proposition], there is an exact sequence of groups (all group schemes are commutative)

$$\begin{aligned} 1 \longrightarrow \mathrm{SO}_2(\mathbb{R}) \longrightarrow \mathcal{G}(\mathbb{R}) \xrightarrow{\det} \mathbb{G}_m(\mathbb{R}) \longrightarrow \\ \xrightarrow{\delta} \mathrm{H}^1(\Gamma, \mathrm{SO}_2(\mathbb{C})) \xrightarrow{1} \mathrm{H}^1(\Gamma, \mathcal{G}(\mathbb{C})) \xrightarrow{j} \mathrm{H}^1(\Gamma, \mathbb{G}_m(\mathbb{C})) \end{aligned}$$

where the "1" comes from the surjectivity of δ. This implies that the map j is injective and thus $\mathrm{H}^1(\Gamma, \mathcal{G}(\mathbb{C}))$ is trivial, since $\mathrm{H}^1(\Gamma, \mathbb{G}_m(\mathbb{C}))$ is trivial. Therefore, no $\mathbb{C}/\mathbb{R}$-forms of R and $\mathcal{G}$ occur in this example.

For the example of $\mathcal{G} \cong \mathcal{G}_a^2 \rtimes \mathrm{SO}_2$, we refer the reader to section 2.4 in the next chapter, since it demonstrates a solution of a unipotent embedding problem.

1.11 K-effective descent

Let $\tilde{R}$ be a Picard-Vessiot ring over $\overline{K}(t)$ with field of constants $\overline{K}$ and Galois group scheme $\tilde{\mathcal{G}}$. Theorem 1.25 gives a sufficient condition for the descent of $\tilde{R}$ to a pseudo Picard-Vessiot ring R over $K(t)$. However, it is not clear which K-structure of $\tilde{\mathcal{G}}$ is realized by R. Next, we look at a special situation in which we can control this.

Theorem 1.38. *Let $\mathcal{G}$ be a connected affine group scheme over K and let $A \in \mathrm{Lie}_{K(t)}(\mathcal{G})$. Suppose that $\tilde{R}$ is a pseudo Picard-Vessiot ring over $\overline{K}(t)$ for the equation defined by A such that $\mathcal{G}_{\overline{K}}$ is the Galois group scheme of $\tilde{R}$ over $\overline{K}(t)$. Then there exists a K-structure R on $\tilde{R}$ which realizes $\mathcal{G}$ as Galois group scheme. In this case we say that $\mathcal{G}$ is K-effectively realizable.*

Proof. First, we use results from classical differential Galois theory in order to replace $\tilde{R}$ by a more concise Picard-Vessiot ring.
Let $K[\mathcal{G}] = K[Z_{ij}, \det(Z)^{-1}]/P$ be the Hopf algebra of $\mathcal{G}$. Then $K[\mathcal{G}] \otimes_K \overline{K}$ is the Hopf algebra of $\mathcal{G}_{\overline{K}}$. Since $\overline{K}(t)$ is flat over K we have

$$\tilde{R}' := K[\mathcal{G}] \otimes_K \overline{K}(t) = \overline{K}(t)[Z_{ij}, \det(Z)^{-1}]/(P \otimes_K \overline{K}(t)).$$

We define a derivation on $\overline{K}(t)[Z_{ij}, \det(Z)^{-1}]$ by $\partial(Z) = AZ$ which induces a derivation on $\tilde{R}'$ since $P \otimes_K \overline{K}(t)$ is a differential ideal (this is due to the fact that $A \in \mathrm{Lie}_{K(t)}(\mathcal{G})$, see the proof of [PS03, p.24, Proposition 1.31(1)]). By Theorem 1.34 and [Ser97, III,2.3 Theorem 1'] we know that $\mathrm{Spec}(\tilde{R})$ is a trivial $\mathcal{G}_{\overline{K}(t)}$-torsor and thus $\tilde{R}' \cong \tilde{R}$. Therefore the ideal $P \otimes_K \overline{K}(t)$ must be a maximal differential ideal and $\tilde{R}'$ is a Picard-Vessiot ring for A (if it was not maximal then $\tilde{R}$ would not be simple). Without loss of generality, we replace $\tilde{R}$ by $\tilde{R}'$. A Γ-action on $\tilde{R}$ is given by $\sigma(Z) = Z$ for all $\sigma \in \Gamma$ since A has Γ-invariant entries (in $K(t)$) and the ideal P is generated by polynomials with coefficients in K. The ring of invariants under this action is $R = K(t)[Z_{ij}]/(P \otimes_K K(t))$.
The Galois group scheme of R is $\mathcal{G}$. To show this, we use the notation of the proof of theorem 1.26 and let B be a K-algebra. The rule

$$Z \mapsto ZX \ , \ X \in \mathrm{GL}_n(B)$$

induces a well defined $K(t)_B$-linear automorphism of R_B if and only if $X \in \mathcal{G}(B)$. The assertion follows. $\square$

1.12 Upper bound by flow

Using a little bit of analysis we can prove the following theorem which gives an upper bound for the Galois group scheme.

Theorem 1.39. *Suppose there exists an embedding of K into the field $\mathbb{C}$ of complex numbers. Let $\mathcal{G}$ be an algebraic affine group scheme over K and let $A \in \mathrm{Lie}_{K(t)}(\mathcal{G})$. Then there exists a pseudo Picard-Vessiot field $E/K(t)$ which is generated by a fundamental matrix $Y \in \mathcal{G}(E)$ and the Galois group scheme $\mathrm{Gal}_D(E/K(t))$ is a closed K-subgroup scheme of $\mathcal{G}$.*

Proof. We use an analytical interpretation of the situation and consider K as a subfield of $\mathbb{C}$. Next, we consider the group $\mathcal{G}(\mathbb{C})$ as a complex Lie group ([OV90, Chapter 3, §1, Theorem 2]).
For each regular value t the matrix A is a vector in the tangent space of $\mathcal{G}(\mathbb{C})$ at the identity element. Multiplication by $g \in \mathcal{G}(\mathbb{C})$ transports A to the vector Ag in the tangent space at g. The map

$$V : g \mapsto Ag$$

defines a smooth vector field on $\mathcal{G}(\mathbb{C})$. In order to simplify the notations we suppose that $t = 0$ is a regular value (otherwise we can achieve this by linear transformation of t). Then the flow equation

$$\frac{d}{dt} y(t) = V(y(t))$$

has a unique smooth solution $y(t)$ on a small intervall $t \in (-a, +a)$ for the initial condition $y(0) = 1$ ([OV90, Chapter 1, Proposition 1]). To this end, the entries of the matrix $y(t)$ may be represented by elements of $\mathbb{C}[[t]]$.
Since $y(t_0) \in \mathcal{G}(\mathbb{C})$ for every $t_0 \in (-a, +a)$ we conclude that $y(t) \in \mathcal{G}(\mathbb{C}[[t]])$ by the identity theorem for power series.
On the other hand we can compute a formal solution of the differential equation $\partial(Y) = AY$. Therefore we let $Y_0 = 1$ and use the equation

$$\partial(Y_0 + Y_1 t + Y_2 t^2 + \dots) = Y_1 + 2Y_2 t + \dots = (A_0 + A_1 t + \dots)(Y_0 + Y_1 t + \dots)$$

in order to compute $Y = Y_0 + Y_1 t + Y_2 t^2 + \dots$ inductively by comparing coefficients. Since A has entries in $K(t)$, it is clear that the coefficient matrices of Y has entries in K. The uniqueness of the solution $y(t)$ implies that $y(t) = Y$ also has coefficients in $K[[t]]$. Thus, we conclude $Y \in \mathcal{G}(K[[t]])$.
It is clear that Y generates a pseudo Picard-Vessiot field $E/K(t)$. Indeed, the field of constants does not change because Y has entries in the field $\mathrm{Quot}(K[[t]])$ which has field of constants K. We represent the Galois group scheme via Y and claim that $\mathrm{Gal}_D(E/K(t))$ is a closed K-subgroup scheme of $\mathcal{G}$. Using the notation of the proof of Theorem 1.26, the group $\mathcal{G}(B)$ for a K-algebra B is defined to be the group of matrices $C \in \mathrm{GL}_n(B)$ such that $Y \mapsto YC$ induces a well defined automorphism of E (the automorphisms of a pseudo Picard-Vessiot field are naturally isomorphic to the automorphisms of its pseudo Picard-Vessiot ring). This implies that YC has to be in $\mathcal{G}(E)$ since $Y \in \mathcal{G}(E)$ and the automorphism has to respect the relations between the entries of Y. Consequently, the matrix C has to be an element of $\mathcal{G}(B)$ and therefore $\mathrm{Gal}_D(E/K(t))(B) \subset \mathcal{G}(B)$. □

The latter theorems allow us to head for inverse problems. Is every affine group scheme $\mathcal{G}$ of finite type over K realized as Galois group scheme of a pseudo Picard-Vessiot field over $K(t)$? We will give a positive answer for connected group schemes over $\mathbb{Q}$ in the next chapter.

Chapter 2

The connected inverse problem over $\mathbb{Q}(t)$

In the previous chapter we used the language of schemes in order to describe the symmetries of differential equations. We did this because this language is natural when working over non algebraically closed fields. In this chapter we will use results from [Spr98]. The book uses the language of algebraic varieties in order to describe linear algebraic groups. We need a way to compare these languages.
This is given by [Har77, II, Proposition 4.10] where a functor from the category of varieties over an algebraically closed field L into the category of schemes over L is constructed. We denote this functor by $\mathcal{F}$. If we restrict $\mathcal{F}$ to the category of linear algebraic groups over L then the image of $\mathcal{F}$ is the subcategory of affine group schemes of finite type over L.
Let K be a subfield of L. A K-structure on a linear algebraic group $\tilde{G}$ over L corresponds to an affine group scheme $\mathcal{G}$ of finite type over K such that $\mathcal{G}_L \cong \mathcal{F}(\tilde{G})$. Thus, the category of linear algebraic groups over L with a fixed K-structure is equivalent to the category of affine group schemes of finite type over K.
This equivalence allows us to use the term "linear algebraic group over K" for actual linear algebraic groups as well as for the above group schemes. Which object is meant will be clear from the context.

Definition 2.1. *Let F be a differential field with field of constants K. We say that a linear algebraic group $\mathcal{G}$ over K is* **realizable over** *F if there exists a pseudo Picard-Vessiot ring R over F whose Galois group scheme is K-isomorphic to $\mathcal{G}$.*

Definition 2.2. *We say that a connected linear algebraic group $\mathcal{G}$ over K is* **K-effectively realizable** *if there exists a pseudo Picard-Vessiot ring R over F for a matrix $A \in \mathrm{Lie}_F(\mathcal{G})$ whose Galois group scheme is K-isomorphic to $\mathcal{G}$. A differential equation is called* **K-effective** *if it induces a K-effective realization of $\mathcal{G}$.*

Remark: Since we consider connected groups in Definition 2.2, it is clear by Theorem 1.32 and Proposition 1.33 that the respective pseudo Picard-Vessiot ring is in fact a Picard-Vessiot ring over F.

Let $\mathcal{G}$ be a connected linear algebraic group over K. By Theorem 1.38 we know that if we can realize $\mathcal{G}_{\overline{K}}$ by a matrix $A \in \mathrm{Lie}_{K(t)}(\mathcal{G}_{\overline{K}})$, then the same matrix induces a

K-effective realization of $\mathcal{G}$. It is known that we can $\overline{K}$-effectively solve the inverse problem over $\overline{K}(t)$, i.e., the connected linear algebraic group $\mathcal{G}_{\overline{K}}$ over $\overline{K}$ is realizable by a matrix $A \in \mathrm{Lie}_{\overline{K}(t)}(\mathcal{G}_{\overline{K}})$. Thus, all we have to do in order to solve the inverse problem over $K(t)$ is to check that we can in fact choose $A \in \mathrm{Lie}_{K(t)}(\mathcal{G})$.

We will give a positive answer to the connected inverse problem over $K(t)$ if K is embeddable into the field $\mathbb{C}$. This property is needed in two steps: the first one is the realization of tori, and the second one is the use of Theorem 1.39 in order to solve embedding problems with unipotent abelian kernel. All other results are proved without this assumption and we conjecture that one can get rid of it completely. Therefore, throughout this chapter K is an arbitrary field of characteristic 0 unless otherwise stated.
Note that since we work in characteristic 0, the field K is infinite. This is relevant for many results from the book [Spr98] applied in this chapter.

2.1 Semisimple groups

The following theorem uses the root space decomposition of the Lie algebra and the notion of a Chevalley module. Details may be found in [PS03] or in the original proof [MS96].

Theorem 2.3. *Let $\mathcal{G} \leq \mathrm{GL}_n(\overline{K})$ be a connected semisimple linear algebraic group over K. Then $\mathcal{G}$ is K-effectively realizable over $K(t)$.*

Proof. We follow the proof of [PS03, p.289, Theorem 11.30]. Thanks to Theorem 1.38, we only need to check that we can choose an effective differential equation with coefficients in $K(t)$.
By [Spr98, 13.3.6], there exists a maximal torus $\mathcal{T} \leq \mathcal{G}$ which is defined over K. Let $\mathfrak{g} = \mathrm{Lie}_{\overline{K}}(\mathcal{G})$ with root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{t} = \mathrm{Lie}_{\overline{K}}(\mathcal{T})$ and Φ is the set of nonzero roots of $\mathfrak{g}$. Let $\rho : \mathcal{G} \longrightarrow \mathrm{GL}(W)$ be a Chevalley module for $\mathcal{G}$ as linear algebraic group over $\overline{K}$ ([PS03, Definition 11.33, Lemma 11.34]) and let χ be the induced representation of $\mathfrak{g}$ on W. The $\overline{K}$-vector space W decomposes into the direct sum of weight spaces

$$W = \bigoplus_{\beta \in \mathfrak{t}^*} W_\beta, \text{ where } \mathfrak{t}^* \text{ is the dual space of } \mathfrak{t}.$$

In the original proof, one chooses a matrix A_0 to be

$$A_0 = \sum_{\alpha \in \Phi} X_\alpha \quad (\mathfrak{g}_\alpha = \overline{K} X_\alpha). \tag{2.1}$$

A second matrix $A_1 \in \mathfrak{t}$ is chosen so that the following conditions hold.

(1) The numbers $\alpha_i(A_1)$ are nonzero and distinct.

(2) The numbers $\beta(A_1)$ are nonzero and distinct for the nonzero weights β of the representation ρ.

(3) If m is an integral eigenvalue of $B := \sum_{\alpha\in\Phi} \frac{1}{\alpha(A_1)} X_\alpha X_{-\alpha}$, then $m = 0$.

The proof proceeds by showing that the matrix $A_0 + A_1 t$ realizes the group $\mathcal{G}$ over $\overline{K}(t)$. Therefore, by Theorem 1.38, it is enough to show that A_0 and A_1 can be chosen with entries in K. Let U denote the set of elements of $\mathfrak{t}$ which satisfy the properties (1) and (2). We consider $\mathfrak{t} \cong \mathbb{A}^r$ as a variety where r is the dimension of $\mathfrak{t}$ as a vector space over $\overline{K}$. The set U is the complement of the Zariski-closed subset $\mathcal{V}(f) \subset \mathfrak{t}$ where

$$f = \prod_i \alpha_i \prod_j \beta_j \prod_{k\neq l}(\alpha_k - \alpha_l) \prod_{s\neq t}(\beta_s - \beta_t).$$

Since f is nontrivial, the set U is a nonempty open subset of the irreducible variety $\mathfrak{t}$ and therefore U is dense in $\mathfrak{t}$. Since $\mathcal{T}$ is defined over K so is $\mathfrak{t}$ and $\mathfrak{t}(K) \subset \mathfrak{t}$ is a dense subset. Suppose that $U \cap \mathfrak{t}(K) = \emptyset$. Then $\mathfrak{t}(K) \subset \mathcal{V}(f)$ and therefore $f(\mathfrak{t}(K)) = 0$. Because f is continuous and $\mathfrak{t}(K) \subset \mathfrak{t}$ is dense, we conclude $f(\mathfrak{t}) = 0$ which is a contradiction to $U \neq \emptyset$.
Thus we can choose a matrix $A_1 \in U \cap \mathfrak{t}(K)$. If property (3) is not fulfilled we can achieve it by multiplying A_1 by an integer as follows. A multiplication of A_1 by $a \in \mathbb{Z}\backslash\{0\}$ induces a multplication of the matrix B in (3) (and its eigenvalues) by a^{-1}. We can therefore choose

$$a > \max\left\{|e| \,\middle|\, e \in \mathbb{Z},\ e \text{ eigenvalue of } B\right\}.$$

The Galois group Γ of $\overline{K}/K$ acts on $\mathfrak{t}^*$ via

$$\sigma(\alpha) := \sigma \circ \alpha \circ \sigma^{-1}$$

for $\sigma \in \Gamma$ and $\alpha \in \mathfrak{t}^*$. This induces an action on Φ since

$$[t, \sigma(X_\alpha)] = \sigma([\sigma^{-1}(t), X_\alpha]) = \sigma(\alpha(\sigma^{-1}(t)))\sigma(X_\alpha) = \sigma(\alpha)(t)\sigma(X_\alpha)$$

for $\alpha \in \Phi$, $X_\alpha \in \mathfrak{g}_\alpha$ and $t \in \mathfrak{t}$. We conclude that $\mathfrak{g}_{\sigma(\alpha)} = \sigma(\mathfrak{g}_\alpha) \neq 0$.
If we define the set

$$M := \{\mathfrak{g}_\alpha |\ \alpha \in \Phi\}$$

then Γ also acts on M. We decompose $M = \cup_i M_i$ into the orbits with respect to this action. Since the subspaces

$$\mathfrak{g}_i := \bigoplus_{\mathfrak{g}_{ij}\in M_i} \mathfrak{g}_{ij}$$

of $\mathfrak{g}$ are Γ-stable, they are defined over K. For each i we choose $C_i \in \mathfrak{g}_i(K)$ with $C_i \neq 0$. We may write

$$C_i = \sum_j C_{ij}$$

with unique $C_{ij} \in \mathfrak{g}_{ij}$. Since $C_i \neq 0$ we may suppose without loss of generality that $C_{i1} \neq 0$. We claim that each C_{ij} is nonzero.
Suppose this is not true. Then there exists an index k such that $C_{ik} = 0$. Since M_i

is an orbit under Γ there exists $\sigma \in \Gamma$ such that $\sigma(\mathfrak{g}_{i1}) = \mathfrak{g}_{ik}$. The group Γ acts via automorphisms. Therefore we have $\sigma(C_{i1}) \neq 0$ which implies that the summand in

$$\sigma(C_i) = \sum_j \sigma(C_{ij})$$

belonging to $\mathfrak{g}_{ik}$ is nonzero. We conclude that $\sigma(C_i) \neq C_i$ which is a contradiction to $C_i \in \mathfrak{g}_i(K)$.
Finally, we define

$$A_0 := \sum_i C_i \quad (\in \mathfrak{g}(K))$$

which clearly satisfies condition (2.1).

□

2.2 Tori

Let L be a Galois extension of K with Galois group Γ. There is an equivalence between the category of K-tori that split over L and the category dual to the category of finitely generated $\mathbb{Z}[\Gamma]$-modules which are free over $\mathbb{Z}$. It is given by associating to a K-torus the module of its group of $\overline{K}$-characters ([Tit68, Chapter III, 1.4.3 Proposition]). Equipped with this tool we will realize every K-torus. First we will realize "indecomposable" tori.

Definition 2.4. *A K-torus $\mathcal{T}$ is called K-indecomposable if there exists no decomposition $\mathcal{T} \cong \mathcal{T}_1 \times \mathcal{T}_2$ with nontrivial K-tori $\mathcal{T}_1, \mathcal{T}_2$.*

Thanks to the following theorem, we can restrict ourselves to finite Galois extensions.

Theorem 2.5. *Every K-torus splits over a finite Galois extension L/K.*

Proof. [Tit68, Chapter III, 1.4.1 Theorem] □

Lemma 2.6. *Let $\mathcal{T}$ be a K-indecomposable K-torus which splits over the finite Galois extension L/K with Galois group Γ. Let X denote the group of characters of $\mathcal{T}$ which are defined over L. Then X is a $\mathbb{Z}[\Gamma]$-submodule of $\mathbb{Z}[\Gamma]$.*

Proof. By [Tit68, Chapter III, 1.6.3 Lemma], X is a submodule of a finitely generated free $\mathbb{Z}[\Gamma]$-module, i.e., there exists a $\mathbb{Z}[\Gamma]$-linear map $\varphi : X \hookrightarrow M$ with $M \cong \mathbb{Z}[\Gamma]^r$ for an integer r. We have to show that we may choose $r = 1$.
Suppose that we chose r to be minimal. If $r > 1$ then we may write $M = M_1 \oplus M_2$ where M_1 and M_2 are nontrivial free submodules of M. Since M_1 and M_2 are free $\mathbb{Z}[\Gamma]$-modules it is clear that they are Γ-stable. Due to the minimality of r, the intersection of $\varphi(X)$ with each M_i is nontrivial. But then $X = \varphi^{-1}(M_1) \oplus \varphi^{-1}(M_2)$ is a nontrivial decomposition of X into Γ-stable submodules. These submodules are free as $\mathbb{Z}$-modules since $\mathbb{Z}$ is a principal ideal domain. By the equivalence of categories mentioned above we conclude that the coproduct decomposition of X corresponds to a product decomposition of $\mathcal{T}$ into nontrivial K-subtori. But this is a contradiction to the K-indecomposability of $\mathcal{T}$. Therefore we have $r = 1$. □

Proposition 2.7. *Let K be embeddable into $\mathbb{C}$. Let $\mathcal{T}$ be a K-indecomposable K-torus which splits over the finite Galois extension L/K and let $r = \dim(\mathcal{T})$. Then there exists an element in $\mathrm{Lie}_K(\mathcal{T})$ whose eigenvalues (which lie in L) span a $\mathbb{Q}$-vectorspace of dimension r.*

Proof. Let $\Gamma = \{\sigma_1, \ldots, \sigma_n\}$ where n is the K-dimension of L. We choose an algebraic closure $\overline{K}$ of K such that $L \subset \overline{K} \subset \mathbb{C}$. We construct a K-torus $\mathcal{H}$ and with character group $\mathbb{Z}[\Gamma]$. Let $\mathcal{H}(K) \cong L^*$. We consider $\mathcal{H}(K)$ as a subgroup of $\mathrm{GL}_n(K)$ by choosing some K-basis of L and mapping $x \in L^*$ to the representation of the homomorphism "multiplication by x" with respect to this basis. The group $\mathcal{H}(K) \subset \mathrm{GL}_n(K)$ is defined by linear equations with coefficients in K. The same equations define the group scheme $\mathcal{H} \leq \mathrm{GL}_n$ over K.
Since the torus $\mathcal{H}$ splits over L, there exists a matrix $C \in \mathrm{GL}_n(L)$ such that

$$C^{-1}xC = diag(x_1, \ldots, x_n), \text{ for every } x \in \mathcal{H}(\overline{K}),$$

where $x_1, \ldots, x_n$ are the eigenvalues of $x \in \mathcal{H}(\overline{K}) \subset \mathrm{GL}_n(\overline{K})$. The homomorphisms

$$\alpha_i(x) := x_i$$

constitute a $\mathbb{Z}$-basis of the character group $\mathbb{Z}[\Gamma]$ of $\mathcal{H}$. The restrictions of the α_i to $\mathcal{H}(K)$ equal σ_i because the eigenvalues of $x \in \mathcal{H}(K)$ are just the Galois conjugates of x. It is clear that the character module of $\mathcal{H}$ equals $\mathbb{Z}[\Gamma]$.

Since $\overline{K} \subset \mathbb{C}$, we may consider $\mathcal{H}$ as Lie group over $\mathbb{C}$. As $\mathrm{Lie}_K(\mathcal{H}) = L$, there exists an element $y \in \mathrm{Lie}_K(\mathcal{H})$ such that the Galois conjugates $\{\sigma_i(y) | \; 1 \leq i \leq n\}$ are linearly independent over K (normal basis). We claim that no nontrivial $\overline{K}$-character of $\mathcal{H}$ vanishes on $\exp(y)$.
Every such character ψ of $\mathcal{H}$ is of the form

$$\psi = \alpha_1^{a_1} \ldots \alpha_n^{a_n}$$

where $a_1, \ldots, a_n \in \mathbb{Z}$. Using the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\psi"] & \overline{K}^* \\
\mathrm{Lie}_{\overline{K}}(\mathcal{H}) \arrow[u, "\exp"] \arrow[r, "\mathrm{d}_1\psi"] & \overline{K} \arrow[u, "\exp"']
\end{tikzcd}$$

we compute

$$\begin{aligned}
\psi(\exp(y)) = \exp(\mathrm{d}_1\psi(y)) &= \exp(a_1\, \mathrm{d}_1\alpha_1(y) + \cdots + a_n\, \mathrm{d}_1\alpha_n(y)) \\
&= \exp(a_1\sigma_1(y) + \cdots + a_n\sigma_n(y)).
\end{aligned} \tag{2.2}$$

The last step is due to the fact that each α_i is linear and that its restriction to $\mathcal{H}(K)$ yields σ_i. More precisely, the set $\mathrm{Lie}_K(\mathcal{H})$ consists of $\mathcal{H}(K)$ and the zero matrix, and therefore the matrix C considered above also diagonalizes $\mathrm{Lie}_K(\mathcal{H})$. But this implies that $\alpha_i = \mathrm{d}_1\alpha_i$ if we restrict the maps to nonzero matrices.
From formula (2.2), we conclude that no nontrivial $\overline{K}$-character of $\mathcal{H}$ vanishes on $\exp(y)$.

It follows immediately from the Lemma 2.6 that there exists a surjective K-homomorphism $\phi:\ \mathcal{H} \longrightarrow \mathcal{T}$.

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\phi"] & \mathcal{T} \\
\mathrm{Lie}_K(\mathcal{H}) \arrow[u, "\exp"] \arrow[r, "\mathrm{d}_1\phi"] & \mathrm{Lie}_K(\mathcal{T}) \arrow[u, "\exp"']
\end{tikzcd}$$

The pullback of characters on $\mathcal{T}$ to characters on $\mathcal{H}$ is injective by Lemma 2.6. Thus, we conclude that no nontrivial $\overline{K}$-character of $\mathcal{T}$ vanishes on $\phi(\exp(y))$.
We claim that $z := \mathrm{d}_1\phi(y)$ is an element with the desired properties.
Without loss of generality we suppose that $\mathcal{T} \subset \mathrm{GL}_m(\overline{K})$ for some m. Again, we may choose a matrix $D \in \mathrm{GL}_m(L)$ such that

$$D^{-1}xD = \mathrm{diag}(x_1, \ldots, x_m), \text{ for every } x \in \mathcal{T}(\overline{K}).$$

The map

$$\chi_i : x \mapsto x_i$$

is a character for each i. It is clear that, after renumeration of the indices, we obtain a $\mathbb{Z}$-basis $\{\chi_1, \ldots, \chi_r\}$ of $X(\mathcal{T})$. By the same considerations we made above for the group $\mathcal{H}$, the elements $\mathrm{d}_1\chi_1(z) = \chi_1(z), \ldots, \mathrm{d}_1\chi_r(z) = \chi_r(z)$ are eigenvalues of z. Suppose that

$$\sum_i \lambda_i \, \mathrm{d}_1\chi_i(z) = 0$$

is a $\mathbb{Q}$-linear relation and, multiplying by denominators, suppose without restriction that $\lambda_i \in \mathbb{Z}$. Applying exp, we conclude that

$$\prod_i \chi_i^{\lambda_i}(\exp(z)) = 1.$$

On the other hand, the commutativity of the above diagram implies

$$\prod_i \chi_i^{\lambda_i}(\exp(z)) = \prod_i \chi_i^{\lambda_i}(\phi(\exp(y))), = \phi^*(\prod_i \chi_i^{\lambda_i})(\exp(y)),$$

where $\phi*$ denotes the pullback via ϕ and so $\lambda_i = 0$ for all i, by injectivity of the pullback. This proves the assertion. □

In the case $K = \mathbb{R}$, the notion of an $\mathbb{R}$-indecomposable $\mathbb{R}$-torus can be made more explicit.

Proposition 2.8. *Let $\mathcal{T}$ be an $\mathbb{R}$-indecomposable $\mathbb{R}$-torus. Then $\mathcal{T}$ is $\mathbb{R}$-isomorphic to either $\mathbb{G}_m$, SO_2 or $\mathcal{H}$, where $\mathcal{H}$ is the group scheme having the character group $\mathbb{Z}[\Gamma]$. Note that we saw $\mathbb{R}$-effective realizations of these schemes in examples (1), (2) and (4).*

Proof. Since we do not use this proposition, we omit the straightforward proof. □

Example: Look at the anisotropic $\mathbb{R}$-torus

$$\mathrm{SO}_2 = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{C} \right\}.$$

Let $\Gamma = \{\mathrm{id}, \tau\}$ denote the Galois group of $\mathbb{C}/\mathbb{R}$. Then Lemma 2.6 tells us that the character group $X = \mathbb{Z}[\chi]$ with

$$\chi(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}) = a + ib$$

is a submodule of $\mathbb{Z}[\Gamma]$. Clearly, $\tau(\chi) = -\chi$ and thus the map $\chi \mapsto \mathrm{id} - \tau$ defines the desired injection. Since the $\mathbb{R}$-torus which belongs to $\mathbb{Z}[\Gamma]$ is the 2-dimensional torus $\mathcal{H}$ from Example (4) in Section 1.10 with $\mathbb{R}$-valued points $\mathbb{C}^*$, the above injection defines a surjective $\mathbb{R}$-homomorphism

$$\phi : \mathbb{C}^* \longrightarrow \mathrm{SO}_2(\mathbb{R}),\ x + iy \mapsto \frac{1}{x^2+y^2} \begin{pmatrix} x^2 - y^2 & -2xy \\ 2xy & x^2 - y^2 \end{pmatrix}$$

which induces the $\mathbb{R}$-linear map

$$\mathrm{d}_1\phi : \mathbb{C} \longrightarrow \mathfrak{so}_2(\mathbb{R}),\ a + ib \mapsto \begin{pmatrix} 0 & -2b \\ 2b & 0 \end{pmatrix}$$

on the Lie algebras. The element $1 + i \in \mathbb{C}$ induces a normal basis over $\mathbb{R}$ and thus the element $\exp(1+i)$ is dense in $\mathbb{C}^*$. Then the element

$$y := \mathrm{d}_1\phi(1+i) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

has the dense exponential

$$\exp(y) = \begin{pmatrix} \cos(2) & -\sin(2) \\ \sin(2) & \cos(2) \end{pmatrix}$$

which implies that the eigenvalues of y span a one dimensional $\mathbb{Q}$-vectorspace (of course this fact is trivial here).

Theorem 2.9. *Let $\mathcal{T}$ be a K-torus. Then $\mathcal{T}$ is K-effectively realizable.*

Proof. We decompose $\mathcal{T}$ into K-irreducible subtori $\mathcal{T}_1, \dots, \mathcal{T}_s$. By Proposition 2.7 we may choose elements $A_1 \in \mathrm{Lie}_K(\mathcal{T}_1), \dots, A_s \in \mathrm{Lie}_K(\mathcal{T}_s)$ such that the eigenvalues have the property from the proposition. We define

$$A := A_1 \oplus tA_2 \oplus \cdots \oplus t^{s-1}A_s$$

which is an element of $\mathrm{Lie}_{K(t)}(\mathcal{T})$ having $\dim(\mathcal{T})$ $\mathbb{Q}$-linearly independent eigenvalues. Over $\overline{K}(t)$ the equation induced by A is equivalent to a diagonal equation. By [MS96, p.24, Proposition 3.7] this diagonal equation $\overline{K}$-effectively realizes the torus $\mathcal{T}$ over $\overline{K}(t)$. But then, by Theorem 1.38, the matrix $A \in \mathrm{Lie}_{K(t)}(\mathcal{T})$ yields a K-effective realization of $\mathcal{T}$. □

2.3 Reductive groups

Theorem 2.10. *Let $\mathcal{G}$ be a connected reductive linear algebraic group over K. Then $\mathcal{G}$ is K-effectively realizable over $K(t)$.*

Proof. By [Spr98, 2.2.8 Corollary] the commutator $\mathcal{G}' := (\mathcal{G}, \mathcal{G})$ is defined over K and semisimple. Therefore, by Theorem 2.3, $\mathcal{G}'$ is K-effectively realizable by $A = A_0 + A_1 t \in \mathrm{Lie}_{K(t)}(\mathcal{G}')$ with $A_0, A_1 \in \mathrm{M}_n(K)$.
The radical $\mathcal{R}(\mathcal{G}) = \mathcal{Z}(\mathcal{G})^o$ is a torus defined over K by [Spr98, 12.1.7(b)]. By Theorem 2.9, $\mathcal{R}(\mathcal{G})$ is K-effectively realizable by a matrix $B \in \mathrm{Lie}_K(\mathcal{R}(\mathcal{G}))$.
Next, by [MS96, Corollary 2.9], the group $\mathcal{G}' \times \mathcal{R}(\mathcal{G})$ is realized by $A \oplus B$. The homomorphism

$$\mu : \mathcal{G}' \times \mathcal{R}(\mathcal{G}) \longrightarrow \mathcal{G}, \quad (x, y) \mapsto xy$$

is defined over K and surjective by [Spr98, 8.1.6(i)]. Thus the differential of this map $\mathrm{d}_1\mu$ is also defined over K. But then by [MS96, Proposition 2.7] (with $U = 1$ and $A_U = 0$) we conclude that $\mathrm{d}_1\mu(A \oplus B) \in \mathrm{Lie}_{K(t)}(\mathcal{G})$ yields a K-effective realization of $\mathcal{G}$. □

2.4 Connected groups

Every connected linear algebraic group $\mathcal{G}$ can be written as a semidirect product $\mathcal{U} \rtimes \mathcal{P}$ where $\mathcal{U}$ is the unipotent radical of $\mathcal{G}$ and $\mathcal{P}$ is reductive. We will give a method for realizing $\mathcal{G}$ starting from a realization of $\mathcal{P}$. For this we need the notion of embedding problems.

Definition 2.11. *Let F be a differential field with field of constants K and let R be a pseudo Picard-Vessiot ring over F with Galois group scheme $\alpha : \mathcal{G} \xrightarrow{\cong} \mathrm{Gal}_D(R/F)$ over K. Let*

$$1 \longrightarrow \mathcal{K} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\beta} \mathcal{G} \longrightarrow 1$$

be an exact sequence of group schemes over K. Then we define the corresponding K**-embedding problem** $\mathcal{E}(\alpha, \beta)$ *to be the question about the existence of a pseudo Picard-Vessiot ring S over F and a monomorphism $\mathrm{Gal}_D(S/F) \xrightarrow{\gamma} \tilde{\mathcal{G}}$ of K-group schemes such that the diagram*

$$\begin{tikzcd}
 & & \mathrm{Gal}_D(S/F) \arrow[r, "\mathrm{res}"] \arrow[d, "\gamma"] & \mathrm{Gal}_D(R/F) \arrow[d, "\alpha", "\cong"'] & \\
1 \arrow[r] & \mathcal{K} \arrow[r] & \tilde{\mathcal{G}} \arrow[r, "\beta"] & \mathcal{G} \arrow[r] & 1
\end{tikzcd}$$

is commutative. The group $\mathcal{K}$ is called the **kernel of the embedding problem**. *The monomorphism γ is called a* **solution** *of $\mathcal{E}(\alpha, \beta)$. If it is an isomorphism then we say that it is a* **proper solution**. *The embedding problem is called* K**-effective** *if R/F is a K-effective pseudo Picard-Vessiot extension. If S/F is effective then γ is called an* K**-effective solution**. *An embedding problem is called a* **Frattini embedding problem** *if $\mathcal{K}$ has no other supplement in $\tilde{\mathcal{G}}$ than $\tilde{\mathcal{G}}$ itself. It is called* K**-split** *if the corresponding exact sequence splits and the section is a K-morphism.*

So far we realized reductive groups K-effectively. A connected linear algebraic group $\mathcal{G}$ over K sits within the split exact sequence

$$1 \longrightarrow \mathcal{U} \longrightarrow \mathcal{G} \xrightarrow{\beta} \mathcal{P} \longrightarrow 1$$

of linear algebraic groups over K where $\mathcal{P}$ is reductive and is called a Levi factor of $\mathcal{G}$. (see [Mos56, p.200, Theorem]). The group $\mathcal{U}$ is a K-subgroup of $\mathcal{G}$ ([Spr98, 12.1.7(d)]) and thus $\mathcal{P}$ is a K-group by [Spr98, 12.2.2]. Let $\alpha : \mathrm{Gal}_D(R/F) \xrightarrow{\cong} \mathcal{P}$ be a K-effective realization of $\mathcal{P}$. We want to realize $\mathcal{G}$ by solving the K-effective embedding problem $\mathcal{E}(\alpha, \beta)$ which is split and has a unipotent kernel.
We define $\mathcal{U}' = (\mathcal{U}, \mathcal{U})$ (which is a K-group by [Spr98, 2.2.8]) and decompose this embedding problem into the embedding problems given by

$$1 \longrightarrow \mathcal{U}/\mathcal{U}' \longrightarrow \mathcal{G}/\mathcal{U}' \longrightarrow \mathcal{P} \longrightarrow 1 \tag{2.3}$$

and

$$1 \longrightarrow \mathcal{U}' \longrightarrow \mathcal{G} \xrightarrow{\varphi} \mathcal{G}/\mathcal{U}' \longrightarrow 1, \tag{2.4}$$

where the embedding problem (2.3) has abelian unipotent kernel and embedding problem (2.4) is an effective Frattini embedding problem.
We will solve the Frattini embedding problem first.

Proposition 2.12. *There exists a K-effective proper solution of the Frattini embedding problem given by (2.4).*

Proof. Suppose that $\mathcal{G}/\mathcal{U}'$ is realized K-effectively by the matrix $A \in \mathrm{Lie}_{K(t)}(\mathcal{G}/\mathcal{U}')$. Since φ is a K-homomorphism it is clear that the differential

$$\mathrm{d}_1\varphi : \mathrm{Lie}_{K(t)}(\mathcal{G}) \longrightarrow \mathrm{Lie}_{K(t)}(\mathcal{G}/\mathcal{U}')$$

is a $K(t)$-linear surjective map. Any matrix $A' \in \mathrm{Lie}_{K(t)}(\mathcal{G})$ such that $\mathrm{d}_1\varphi(A') = A$ will induce a K-effective realization of $\mathcal{G}$ (see [Obe03, Bemerkung 3.3.5]). $\square$

Next we further decompose the embedding problem (2.3). We have to pay attention here.

Definition 2.13. *A K-embedding embedding problem given by the sequence*

$$1 \longrightarrow \mathcal{K} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\beta} \mathcal{G} \longrightarrow 1$$

is said to have K-minimal kernel if there are no normal K-subgroups of $\mathcal{K}$ which are also normal in $\tilde{\mathcal{G}}$.

Note that even if $\mathcal{K}$ is K-minimal, there may be $\overline{K}$-subgroups of $\mathcal{K}$ which are also normal in $\tilde{\mathcal{G}}$. This may be seen in the example at the end of this section.

Proposition 2.14. *The embedding problem (2.3) decomposes into embedding problems with K-split K-minimal abelian unipotent kernel.*

Proof. Since the embedding problem (2.3) has reductive cokernel, every representation is fully reducible. Thus, we may decompose it into embedding problems with K-minimal (abelian unipotent) kernel. $\square$

For the next theorem we need the following lemma.

Lemma 2.15. *Let $\tilde{E}/\overline{K}(t)$ be a Picard-Vessiot extension with K-structure $E/K(t)$ and let $w \in E \setminus \{0\}$. Then there exists $c \in K$ such that the equation*

$$\partial(x) = \frac{w}{t-c}$$

has no solution in $\tilde{E}$.

Proof. We need a slightly better version of Satz 3.5.6 in [Obe03]. This lemma assures the existence of an element $c \in \overline{K}$ with the above properties. But it is clear that we can even choose $c \in K$. In the first line of the proof of this theorem, we choose T to be a transcendence basis of K over $\mathbb{Q}$ (which is also a transcendence basis of $\overline{K}$ over $\mathbb{Q}$). Then we have $\mathbb{Q}(T) \subset K$ and since the element c is chosen to be an element of $\mathbb{Q}(T)$ this proves our assertion. $\square$

Theorem 2.16. *Let K be embeddable into the field $\mathbb{C}$ of complex numbers. Then every split K-effective K-embedding problem with K-minimal unipotent kernel is K-effectively solvable over $K(t)$.*

Proof. We consider all groups as linear algebraic groups over $\overline{K}$ which are defined over K. The assertion is proved for $\overline{K}$-minimal kernel ([Obe03]). The K-minimal kernel $\mathcal{A}$ may split into $\overline{K}$-minimal kernels but we ignore this fact and try to realize $\mathcal{A}$ "at once". We mimic the proof in [Obe03] with $\mathcal{A}$ instead of a $\overline{K}$-minimal kernel. Suppose the embedding problem is given by the split exact sequence

$$1 \longrightarrow \mathcal{A} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\beta} \mathcal{G} \longrightarrow 1$$

of K-groups. Let γ be a section of β. Suppose that $A \in \mathrm{Lie}_{K(t)}(\gamma(\mathcal{G}))$ realizes $\gamma(\mathcal{G})$ ($\cong \mathcal{G}$) via the pseudo Picard-Vessiot extension $E/K(t)$ generated by the fundamental matrix $Y \in \gamma(\mathcal{G}(E))$. Let $\tilde{E} = E_{\overline{K}}$. Due to [Spr98, 14.3.5], there exists a K-isomorphism

$$\phi : \mathcal{A} \longrightarrow \mathbb{G}_a^m$$

with differential $\mathrm{d}_1\phi$. The conjugation on $\mathcal{A}$ with an element of $\gamma(\mathcal{G})$ yields a linear automorphism of $\mathbb{G}_a^m \cong \overline{K}^m$ (via ϕ). Let

$$\rho : \gamma(\mathcal{G}) \longrightarrow \mathrm{GL}_m$$

be the corresponding representation which is a K-homomorphism.
Next we claim the existence of an element $v \in \mathbb{G}_a^m(K(t))$ such that the inhomogeneous equation

$$\partial(x) = \rho(Y)^{-1}v$$

has no solution in $\tilde{E}^m$. To this end, we choose an element $\tilde{v} \in K^m \setminus \{0\}$. Thus, at least one coordinate of the vector $\rho(Y)^{-1}\tilde{v}$ is nonzero. But then Lemma 2.15 implies that we can find $c \in K$ such that

$$\partial(x) = \frac{1}{t-c}\rho(Y)^{-1}\tilde{v}$$

has no solution in $\tilde{E}^m$. We set $v = \frac{\tilde{v}}{t-c} \in \mathbb{G}_a^m(K)$.
Suppose that there exists $\tilde{x} \in \tilde{E}^m$ such that

$$\partial(\tilde{x}) = \mathrm{d}_1\rho(A)\tilde{x} + v. \tag{2.5}$$

Then we define $x = \rho(Y)^{-1}\tilde{x} \in \tilde{E}^m$ and compute

$$\begin{aligned}
\partial(x) &= \partial(\rho(Y)^{-1}\tilde{x}) = \partial(\rho(Y)^{-1})\tilde{x} + \rho(Y)^{-1}\partial(\tilde{x})\\
&= -\rho(Y)^{-1}\partial(\rho(Y))\rho(Y)^{-1}\tilde{x} + \rho(Y)^{-1}\,\mathrm{d}_1\rho(A)\tilde{x} + \rho(Y)^{-1}v\\
&= -\rho(Y)^{-1}\,\mathrm{d}_1\rho(A)\tilde{x} + \rho(Y)^{-1}\,\mathrm{d}_1\rho(A)\tilde{x} + \rho(Y)^{-1}v && \text{by Lemma 2.18 below}\\
&= \rho(Y)^{-1}v
\end{aligned}$$

which is a contradiction to the choice of v. We conclude that the equation (2.5) has no solution with coordinates in $\tilde{E}$. Let $\tilde{y}$ be a solution of this equation with coordinates in some Picard-Vessiot extension $\tilde{M}$ of $\tilde{E}$ and let $\tilde{Y} = \phi_{\tilde{M}}^{-1}(\tilde{y}) \in \mathcal{A}(\tilde{M})$ as well as $B = \mathrm{d}_1\phi^{-1}(v) \in \mathrm{Lie}_{K(t)}(\mathcal{A})$.
In order to do the next calculation it is of technical advantage to write the group $\mathbb{G}_a^m$ multiplicatively. This can be achieved by using matrices of the form

$$\begin{pmatrix}
1 & a_1 & a_2 & \cdots & a_m\\
0 & 1 & 0 & \cdots & 0\\
0 & 0 & \ddots & & 0\\
 & & \vdots & & \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}$$

where $(a_1, \dots, a_m) \in \mathbb{G}_a^m$. We immediately see that

$$\partial(u)w = \partial(u) \tag{2.6}$$

for $u, w \in \mathbb{G}_a^m(\tilde{M})$. We consider the multiplicative representation suppressing the introduction of new notation and compute

$$\begin{aligned}
\mathrm{d}_1\phi(B) &= v = \partial(\phi(\tilde{Y})) - \mathrm{d}_1\rho(A)\phi(\tilde{Y})\\
&= \rho(Y)(\rho(Y)^{-1}\partial(\phi(\tilde{Y})) - \rho(Y)^{-1}\,\mathrm{d}_1\rho(A)\phi(\tilde{Y}))\\
&= \rho(Y)(\rho(Y)^{-1}\partial(\phi(\tilde{Y})) + \partial(\rho(Y)^{-1})\phi(\tilde{Y}))\\
&= \rho(Y)(\partial(\phi(Y^{-1}\tilde{Y}Y)))\\
&= \rho(Y)(\partial(\phi(Y^{-1}\tilde{Y}Y))\phi(Y^{-1}\tilde{Y}Y))^{-1} && \text{by (2.6)}\\
&= \rho(Y)\,\mathrm{d}_1\phi(\lambda(Y^{-1}\tilde{Y}Y))\phi(Y^{-1}\tilde{Y}Y))\phi(Y^{-1}\tilde{Y}Y))^{-1} && \text{by Lemma 2.18 below}\\
&= \mathrm{d}_1\phi(Y\lambda(Y^{-1}\tilde{Y}Y)Y^{-1})
\end{aligned}$$

where λ is the logarithmic derivative, i.e., $\lambda(X) = \partial(X)X^{-1}$. Then

$$B = Y\lambda(Y^{-1}\tilde{Y}Y)Y^{-1}$$

since $\mathrm{d}_1\phi$ is an isomorphism. Furthermore we have

$$\begin{aligned} B &= Y\lambda(Y^{-1}\tilde{Y}Y)Y^{-1} = Y(\partial(Y^{-1}Y) + Y^{-1}\lambda(\tilde{Y}Y)Y)Y^{-1} \\ &= -A + \lambda(\tilde{Y}Y) \end{aligned}$$

which proves

$$\partial(\tilde{Y}Y) = (A+B)\tilde{Y}Y.$$

Thus we may suppose that $\tilde{M}/\tilde{E}$ is a Picard-Vessiot extension generated by the fundamental matrix $\tilde{Y}Y$. Since $B \in \mathrm{Lie}_{K(t)}(\mathcal{A}) = \mathrm{Kern}(\mathrm{d}_1\beta)$ we conclude that $\mathrm{d}_1\beta(A+B) = \mathrm{d}_1\beta(A)$ which implies that $A+B$ yields a solution of the embedding problem.
Note, that we do not (yet) know if the equation defined by $A+B$ is effective and therefore we can not apply Theorem 1.38. Thus, we employ Lemma 2.17 below which is an improved version of Theorem 1.39. It is shown that there exist K-structures $K(t) \subset E \subset M$ on $\overline{K}(t) \subset \tilde{E} \subset \tilde{M}$ such that we obtain a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{A} \arrow[r] & \tilde{\mathcal{G}} \arrow[r, "\beta"] & \mathcal{G} \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{Gal}_D(M/E) \arrow[r] \arrow[u, hook] & \mathrm{Gal}_D(M/K(t)) \arrow[r, "\mathrm{res}"] \arrow[u, hook] & \mathrm{Gal}_D(E/K(t)) \arrow[r] \arrow[u, "\cong"] & 1
\end{tikzcd}$$

where all the groups and morphisms are defined over K. The Galois correspondence tells us that $\mathrm{Gal}_D(M/E) \trianglelefteq \mathrm{Gal}_D(M/K(t))$. Furthermore, since $\mathcal{A}$ is commutative, we know that $\mathrm{Gal}_D(M/E) \trianglelefteq \mathcal{A}$. Looking at the above diagram we see that $\mathcal{G}$ and $\mathcal{A}$ generate $\tilde{\mathcal{G}}$ which implies $\mathrm{Gal}_D(M/E) \trianglelefteq \tilde{\mathcal{G}}$. But then, by K-minimality of $\mathcal{A}$, we must have $\mathrm{Gal}_D(M/K(t)) \cong \tilde{\mathcal{G}}$ and therefore the extension is K-effective. $\square$

Lemma 2.17. *The diagram in the proof of the Theorem 2.16 consists of K-groups and K-homomorphisms.*

Proof. We generalize the construction in the proof of Theorem 1.39. Namely, we want to find K-structures of the Picard-Vessiot fields in the tower

$$\overline{K}(t) \subset E_{\overline{K}} \subset M_{\overline{K}} \subset \mathrm{Quot}(\overline{K}[[t]])$$

which fit together. We choose a regular point for the matrices A and $A+B$ in the proof of Theorem 2.16. We first obtain a pseudo Picard-Vessiot field $E/K(t)$ by solving the flow defined by A and next the pseudo Picard-Vessiot extension M/E by solving the flow defined by $A+B$, using the technique from Theorem 1.39.
Thus we obtain a tower of field extensions

$$K(t) \subset E \subset M \subset \mathrm{Quot}(K[[t]])$$

which yields

$$\overline{K}(t) \subset E_{\overline{K}} \subset M_{\overline{K}} \subset \mathrm{Quot}(\overline{K}[[t]])$$

after base extension. The exact sequence

$$1 \longrightarrow \mathcal{A} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\beta} \mathcal{G} \longrightarrow 1$$

of K-groups is given by definition. The sequence

$$1 \longrightarrow \mathrm{Gal}_D(M/E) \longrightarrow \mathrm{Gal}_D(M/K(t)) \xrightarrow{\mathrm{res}} \mathrm{Gal}_D(E/K(t)) \longrightarrow 1$$

of K-groups is exact and consists of K-homomorphisms by the Galois correspondence (1.37). The group $\mathrm{Gal}_D(M/K(t))$ is K-isomorphic to $\mathcal{G}$ by assumption and $\mathrm{Gal}_D(M/K(t))$ is a K-subgroup of $\tilde{\mathcal{G}}$ by Theorem 1.39. We thus obtain the diagram of K-groups

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{A} \arrow[r] & \tilde{\mathcal{G}} \arrow[r, "\beta"] & \mathcal{G} \arrow[r] & 1 \\ 1 \arrow[r] & \mathrm{Gal}_D(M/E) \arrow[r] & \mathrm{Gal}_D(M/K(t)) \arrow[r, "\mathrm{res}"] \arrow[u, hook] & \mathrm{Gal}_D(E/K(t)) \arrow[r] \arrow[u, "\cong"] & 1 \end{tikzcd}$$

which is commutative by [Har02, Proposition 2.12] (we just forget about the K-structures on the groups). Therefore we also obtain a K-morphism $\mathrm{Gal}_D(M/E) \longrightarrow \mathcal{A}$ completing the commutative diagram of the Theorem. □

Lemma 2.18. *Let $\mathcal{G} \xrightarrow{\phi} \mathcal{H}$ be a homomorphism of linear algebraic groups over K and let E be a differential field with field of constants K. Suppose $\mathcal{G} \leq \mathrm{GL}_n$ and $\mathcal{H} \leq \mathrm{GL}_m$. Let $A \in \mathrm{Lie}_E(\mathcal{G})$ and $Y \in \mathcal{G}(E)$ such that $\partial(Y) = AY$. Then*

$$\partial(\phi(Y)) = \mathrm{d}_1\phi(A)\phi(Y).$$

Proof. We work over the dual numbers $E[\varepsilon]$ with $\varepsilon^2 = 0$. Then we have

$$\mathrm{Lie}_E(\mathcal{G}) = \{B \in \mathrm{M}_n(E) | 1 + \varepsilon B \in \mathcal{G}(E[\varepsilon])\}$$

and we also note that this is a special case of

$$\mathrm{T}_C(\mathcal{G})(E) = \{B \in \mathrm{M}_n(E) | C + \varepsilon B \in \mathcal{G}(E[\varepsilon])\}$$

where $C \in \mathcal{G}(E)$ and T_C denotes the tangent space at the point C.
With this in mind we compute

$$\begin{aligned} \phi((1+\varepsilon A)Y) &= \phi(Y + \varepsilon AY) \\ &= \phi(Y) + \varepsilon\, \mathrm{d}_Y\phi(AY) \end{aligned}$$

as follows easily from the Taylor series expansion in direction AY at the point Y. On the other hand we have

$$\begin{aligned} \phi((1+\varepsilon A)Y) &= \phi(1+\varepsilon A)\phi(Y) = (1 + \varepsilon\, \mathrm{d}_1\phi(A))\phi(Y) \\ &= \phi(Y) + \varepsilon\, \mathrm{d}_1\phi(A)\phi(Y) \end{aligned}$$

and conclude that we must have

$$\mathrm{d}_Y\phi(AY) = \mathrm{d}_1\phi(A)\phi(Y).$$

Next by the chain rule we have

$$\partial(\phi(Y)) = \mathrm{d}_Y\phi(\partial(Y)) = \mathrm{d}_Y\phi(AY)$$

which proves the claim. □

Now we can prove the main theorem in this chapter.

Theorem 2.19. *Let K be embeddable into the field $\mathbb{C}$ of complex numbers. Let $\mathcal{G}$ be a connected linear algebraic group over K. Then $\mathcal{G}$ is K-effectively realizable.*

Proof. This is clear by using the tools developed in the previous sections via embedding problems. □

To give a feeling of what is going on in the proof of Theorem 2.16 we construct an example in the case $K = \mathbb{R}$.

Example: $\mathcal{G} \cong \mathbb{G}_a^2 \rtimes \mathrm{SO}_2$
We choose the representation

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix} \mid a, b \in \mathbb{C}\,, a^2 + b^2 = 1 \right\}$$

of SO_2 as well as the representation

$$\left\{ \begin{pmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid c, d \in \mathbb{C} \right\}$$

of $\mathbb{G}_a^2$ as closed subgroups of GL_3. We compute

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix} \begin{pmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix} = \begin{pmatrix} 1 & ac - bd & ad + bc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and conclude that, identifying $\mathbb{G}_a^2$ with $\mathbb{C}^2$, the conjugation of SO_2 on $\mathbb{G}_a^2$ translates into the linear representation

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix}$$

of SO_2. We have the exact sequence

$$1 \longrightarrow \mathbb{G}_a^2 \longrightarrow \mathcal{G} \longrightarrow \mathrm{SO}_2 \longrightarrow 1$$

of $\mathbb{R}$-groups and we may give an $\mathbb{R}$-effective realization of the group SO_2 (considered as subgroup of $\mathcal{G}$) by the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

with corresponding fundamental matrix

$$Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}$$

equipped with the usual differentiation as analytic functions. This is clear from Example (2) in Section 1.10 and defines our embedding problem. Let $\tilde{E}$ be the Picard-Vessiot field generated by Y over $\mathbb{C}(t)$. Note that over $\mathbb{C}$ the space $\mathbb{C}^2 \cong \mathbb{G}_a^2$ decomposes into the SO_2-stable subspaces generated by the vectors

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

respectively. This implies that the unipotent kernel $\mathbb{G}_a^2$ of the embedding problem is $\mathbb{R}$-minimal but not $\mathbb{C}$-minimal.
We use the notation of the proof of Theorem 2.16. In order to solve the embedding problem we have to find $v \in \mathbb{R}(t)^2$ such that the equation

$$\partial(x) = \rho(Y)^{-1} v = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

has no solution in $\tilde{E}$. We choose

$$v = \begin{pmatrix} 0 \\ \frac{1}{t} \end{pmatrix}$$

knowing that the equations

$$\partial(y) = \frac{\sin(t)}{t} \quad \text{and} \quad \partial(y) = \frac{\cos(t)}{t}$$

have no solutions in $\tilde{E}$. Two special solutions of these equations are called the Sine and Cosine integral and can be described analytically. We denote them by $\mathrm{Si}(t)$ and $\mathrm{Ci}(t)$ respectively.
Next we look at the equation

$$\partial(\tilde{x}) = \mathrm{d}_1\rho(A)\tilde{x} + v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ \frac{1}{t} \end{pmatrix}$$

which has the solution

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \mathrm{Si}(t)\cos(t) - \mathrm{Ci}(t)\sin(t) \\ \mathrm{Si}(t)\sin(t) + \mathrm{Ci}(t)\cos(t) \end{pmatrix}$$

and define

$$\tilde{Y} = \phi^{-1}(y) = \begin{pmatrix} 1 & y_1 & y_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as well as

$$B = \mathrm{d}_1\phi^{-1}(v) = \begin{pmatrix} 0 & 0 & \frac{1}{t} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that for

$$\tilde{Y}Y = \begin{pmatrix} 1 & \mathrm{Si}(t) & \mathrm{Ci}(t) \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}$$

we have

$$\partial(\tilde{Y}Y) = (A+B)\tilde{Y}Y.$$

We conclude that the Picard-Vessiot extension $\mathbb{C}(t)[\cos(t), \sin(t), \mathrm{Ci}(t), \mathrm{Si}(t)]$ over $\mathbb{C}(t)$ has the $\mathbb{R}$-structure $\mathbb{R}(t)[\cos(t), \sin(t), \mathrm{Ci}(t), \mathrm{Si}(t)]$ which is an $\mathbb{R}$-effective realization of $\mathcal{G}$.

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Acknowledgements

First and foremost, I would like to thank my supervisor B. Heinrich Matzat for suggesting this nice and interesting subject as well as for his guidance and support throughout the development of this thesis. Moreover, I would like to thank him for the friendly integration into his research group which had a highly motivating effect on me.

My special thanks go to Julia Hartmann for all the inspiring discussions which so often ended up in late-night pizza sessions at the institute. I am also very grateful to her for patiently reading this thesis and giving me critical comments.