Tutorial
on
Differential Galois Theory I

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Motivation

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$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta \qquad \text{polar coordinates}$$

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&= \pi
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algebraically:

- splitting field $\mathbb{Q}(i, \sqrt[4]{2})/\mathbb{Q}$ encodes all relations
- therefore the group $G = \mathrm{Aut}(\mathbb{Q}(i, \sqrt[4]{2})/\mathbb{Q})$ describes the symmetries

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- differential field $E/\mathbb{C}(z)$ generated by all solutions encodes the relations
- Therefore the group $G = \mathrm{Aut}^{\partial}(E/\mathbb{C}(z))$ describes the symmetries.

Picard-Vessiot extensions

Given

- a ∂-field F with constants $\mathbb{C}$, example: $\mathbb{C}(z)$ with $\partial = \frac{d}{dz}$
- an equation $p(y) = a_n\partial^n(y) + \ldots + a_1\partial(y) + a_0 y = 0$ with $a_i \in F$

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A ∂-field E/F is called Picard-Vessiot extension for p

$\Leftrightarrow$ The solutions of p in E form an n-dimensional $\mathbb{C}$-vector space V which generates E/F as a ∂-field.

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Theorem

$G \subset \mathrm{GL}(V)$ is a linear algebraic group.

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- Galois group $G = \mathbb{G}_{\mathrm{m}}(\mathbb{C}) = \mathrm{GL}(V)$

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- Galois group $G = \mathbb{G}_a(\mathbb{C}) = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\} \subset \mathrm{GL}(V)$

Examples over $\mathbb{C}(z)$

$y' = \frac{1}{nz} y$

- Solution space $V = \mathbb{C}\sqrt[n]{z}$
- Picard-Vessiot extension $E = \mathbb{C}(\sqrt[n]{z})$
- ∂-automorphisms: $\sqrt[n]{z} \mapsto \xi\sqrt[n]{z}$
- Galois group $G = \mu_n(\mathbb{C}) \subset \mathrm{GL}(V)$

Galois correspondence

Theorem

For E/F Picard-Vessiot, there is a correspondence

$$\left\{ \begin{array}{c} \textit{closed subgroups} \\ \textit{of } \mathrm{Gal}(E/F) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{intermediate } \partial\textit{-fields} \\ \textit{of } E/F \end{array} \right\}$$

with the usual features.

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- The Kummer exact sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1$$

corresponds to the field tower

$$\overbrace{\underbrace{\mathbb{C}(z) \quad \subset \quad \mathbb{C}(z, \exp(nz))}_{\mathbb{G}_m} \quad \subset \quad \mathbb{C}(z, \exp(z))}^{\mathbb{G}_m}$$

$$\mathbb{C}(z, \exp(nz)) \subset \mathbb{C}(z, \exp(z)) : \mu_n$$

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7. Solution space: $V = \mathbb{C}1 \oplus \mathbb{C}\ln(z-1) \oplus \mathbb{C}\,\mathrm{dilog}(z)$

Our motivation was. . .

$$\text{Is} \quad \int e^{-x^2}\,\mathrm{d}x \quad \text{an elementary function?}$$

$$\Updownarrow$$

$$\text{How symmetric is} \quad y'' + 2xy' = 0 \quad ?$$

Question:

How do we translate the property of being elementary into a statement about symmetries?

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This induces a filtration of the Picard-Vessiot extension E/F

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By Galois correspondence:

$\Rightarrow$ $\mathrm{Gal}(E/F)$ is solvable.

Structure of solvable extensions

Theorem (Kolchin)

$G^o \subset \mathrm{GL}_n(\mathbb{C})$ *connected and solvable then*

$$G^o \cong U \rtimes (\mathbb{G}_\mathrm{m})^r \quad \textit{with } U \textit{ unipotent.}$$

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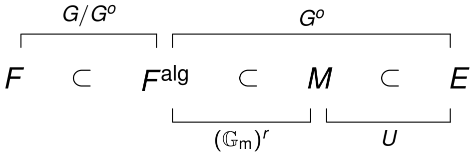

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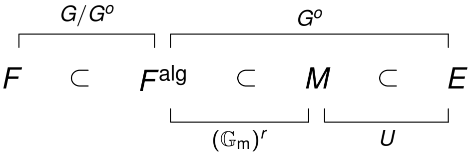

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$$\overbrace{F \subset F^{\mathrm{alg}}}^{G/G^o} \subset M \subset E, \qquad \overbrace{F^{\mathrm{alg}} \subset M \subset E}^{G^o}, \qquad \underbrace{F^{\mathrm{alg}} \subset M}_{(\mathbb{G}_\mathrm{m})^r}, \qquad \underbrace{M \subset E}_{U}$$

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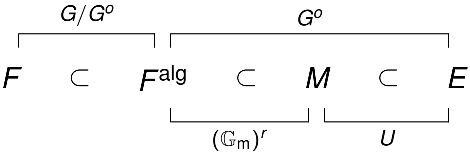

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Conclusion: elementary solutions $\Leftrightarrow$ G^o is abelian

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We have

$$\mathbb{C}(z) \quad \subset \quad \underbrace{\mathbb{C}(z, e^{-z^2})}_{} \quad \subset \quad E$$

$$\underbrace{\mathbb{C}(z) \subset \mathbb{C}(z, e^{-z^2})}_{\mathbb{G}_m} \qquad \underbrace{\mathbb{C}(z, e^{-z^2}) \subset E}_{\mathbb{G}_a}$$

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We compute two automorphisms of $E/\mathbb{C}(z)$:

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$$\underbrace{\mathbb{C}(z) \quad \subset \quad \mathbb{C}(z, e^{-z^2})}_{\mathbb{G}_m} \quad \underbrace{\subset \quad E}_{\mathbb{G}_a}$$

We compute two automorphisms of $E/\mathbb{C}(z)$:

$$\phi_1 : \begin{cases} e^{-z^2} & \mapsto & \lambda e^{-z^2} \\ \int e^{-z^2} & \mapsto & \lambda \int e^{-z^2} \end{cases} \rightsquigarrow \phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\phi_2 : \begin{cases} e^{-z^2} & \mapsto & e^{-z^2} \\ \int e^{-z^2} & \mapsto & \int e^{-z^2} + a \end{cases} \rightsquigarrow \phi_2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

They do not commute.

Approaches to Differential Galois Theory (char. 0)

Basic objects: vector bundles with flat connection $(\mathcal{E}, \nabla)$ on some space X.

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Gauge theoretic approach: Picard-Vessiot theory

- X is an algebraic curve and $(\mathcal{E}, \nabla)$ is usually studied over the generic point of X
- differential Galois group = minimal reduction of the structure group of the frame bundle of $(\mathcal{E}, \nabla)$

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Basic objects: vector bundles with flat connection $(\mathcal{E}, \nabla)$ on some space X.

Singularity theory: Monodromy and Stokes phenomena

- X is a Riemann surface with a finite set of marked points S and $(\mathcal{E}, \nabla)$ is a meromorphic connection with singularities in S.
- differential Galois group = generated by monodromy and Stokes transformations attached to the singularities of $(\mathcal{E}, \nabla)$

Approaches to Differential Galois Theory (char. 0)

Basic objects: vector bundles with flat connection $(\mathcal{E}, \nabla)$ on some space X.

Categorical approach: Tannakian theory

- X is any reasonable space and (Conn/X) is the category of flat vector bundles on X.
- (absolute) differential Galois group = fundamental group of the Tannakian category (Conn/X)