Tutorial on Differential Galois Theory II

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Outline

Today's plan

- Picard-Vessiot rings
- The ∂-Galois group scheme
- The Torsor theorem and applications
- Descent theory for Picard-Vessiot extensions

Systems of ∂-equations

Yesterday we considered:

- a field F with derivation ∂
- an equation $\partial^n(y) + \cdots + a_1\partial(y) + a_0 y = 0$ with $a_i \in F$

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n-th order equation $\Rightarrow$ a system of 1-st order equations:

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$\Rightarrow$ We develop Picard-Vessiot theory for general systems of 1-st order equations:

$$\partial(y) = Ay \quad \text{with } A \in F^{n \times n}$$

which we denote by $[A]$.

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Definition

A Picard-Vessiot ring for $[A]$ is a ∂-ring R/F with

1. R/F is generated by a fundamental solution matrix: $\exists Y \in \mathrm{GL}_n(R) \; : \; \partial(Y) = AY$ and $R = F[Y_{ij}, \det(Y)^{-1}]$
2. R is an integral domain
3. R/F is geometric, i.e. $\mathrm{Quot}(R)$ has no new constants
4. R is ∂-simple, i.e. no non-trivial ∂-ideals

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- Picard-Vessiot field = field of fractions of Picard-Vessiot ring
- coincides with yesterday's definition

Example over $\mathbb{R}(t)$

$y'' + y = 0$

- the 2-nd order equation translates into the system

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- $R = \mathbb{R}(t)[\cos(t), \sin(t)]$ with

$$\cos(t)^2 + \sin(t)^2 = 1$$

  is a Picard-Vessiot ring over $\mathbb{R}(t)$

Geometric interpretation

Dictionary

Differential Algebra	"Differential" Geometry
R ring	Spec(R) affine variety

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non-trivial ∂-ideals	Subvarieties
$(x^2 + y^2 - c)$ with $c \in \mathbb{R}$	concentric circles
(x, y)	origin

Construction of Picard-Vessiot rings

Theorem

Let $A \in F^{n \times n}$ and assume the field of constants K of F is algebraically closed. Then there exists a Picard-Vessiot ring R/F for $[A]$.

Proof.

1. Consider $F[\mathrm{GL}_n] = F[X_{ij}, \det(X)^{-1}]$.

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7. By hypothesis $K = \overline{K} \Rightarrow R/F$ Picard-Vessiot

Basic Picard-Vessiot theory

Our goal is to establish the two main pillars of Picard-Vessiot theory:

- The Galois group G is a linear algebraic group.
- The scheme $\mathrm{Spec}(R)$ is a G-torsor (principal G-bundle).

Schemes à la Grothendieck

- Let $\mathcal{C}$ be a category.
- To an object $X \in \mathcal{C}$ we can associate the functor

$$h_X : \mathcal{C}^{\text{op}} \to (\text{Sets}), Y \mapsto \text{Hom}(Y, X)$$

- The collection of functors $F : \mathcal{C}^{\text{op}} \to (\text{Sets})$ forms a category which we denote by $(\text{Sets})^{\mathcal{C}^{\text{op}}}$.

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Lemma (Yoneda)

The functor

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Definition

Functors which are isomorphic to h_X for some object $X \in \mathcal{C}$ are called representable.

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Applying this to the category of schemes Grothendieck suggests the following

Strategy

The construction of a scheme can be divided into two steps:

1. Construct a functor.
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The Galois group functor

Given a Picard-Vessiot ring R/F with constants K, we define the Galois group functor $\underline{\mathrm{Gal}}(R/F)$

$$(K\text{-algebras}) \to (\text{Groups}), \quad L \mapsto \mathrm{Aut}^{\partial}(R \otimes_K L / F \otimes_K L)$$

Representing functors

Lemma

Assume R/F and R'/F are Picard-Vessiot rings for the same equation $[A]$. Let K be the field of constants in F. Define U to be the K-algebra of constants in $R' \otimes_F R$. Then the map

$$R' \otimes_K U \to R' \otimes_F R, \quad r' \otimes u \mapsto (r' \otimes 1)u$$

is an R'-linear ∂-isomorphism.

Representing functors

Fundamental Lemma: $R' \otimes_K U \cong R' \otimes_F R$

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The functor $\underline{\mathrm{Isom}}^{\partial}_F(R, R')$

$$(K\text{-algebras}) \to (\mathrm{Sets}), \quad L \mapsto \mathrm{Isom}^{\partial}_{F\otimes_K L}(R \otimes_K L, R' \otimes_K L)$$

is represented by the scheme $\mathrm{Spec}(U)$*.*

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Corollary

If the field of constants K is algebraically closed, then Picard-Vessiot rings are unique.

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Corollary

If the field of constants K is algebraically closed, then Picard-Vessiot rings are unique.

Proof. The scheme $\underline{\mathrm{Isom}}^{\partial}(R, R')$ is of finite type over K and therefore has a K-rational point (Hilbert Nullstellensatz).

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Proof. Put $R' = R$.

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One checks:

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- the fact that (1) is an isomorphism is equivalent to

Theorem (Torsor theorem)

Let R/F be a Picard-Vessiot ring with Galois group G. Then $\mathrm{Spec}(R)$ *is a G-torsor over F.*

The Torsor theorem in everyday life

Example

Let R/F be a Picard-Vessiot ring with ∂-Galois group $\mathbb{G}_a$.

1. *Torsor theorem*: Spec(R) is a $\mathbb{G}_a$-torsor over F

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5. $R \cong F[x]$ with $\mathbb{G}_a$ acting via $x \mapsto x + c$

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$\Rightarrow$ R/F is generated by an integral

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3. $\frac{\partial(x)}{x}$ is invariant under $\mathbb{G}_{\mathrm{m}}$

$\Rightarrow$ R/F is generated by an exponential

Other consequences of the Torsor theorem

- Dimension of $G =$ Transcendence degree of E/F
- The scheme Spec(R) is smooth over F
- The Galois correspondence is a rather straightforward consequence of
  1. Torsor theorem
  2. flat descent of quasi-projective schemes
  3. Hilbert 90 with GL_n coefficients

Picard-Vessiot theory only if $K = \overline{K}$?

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$\Rightarrow$ Galois descent provides a method to handle this situation.

Descent

Assume X is a space and $\{U_i\}$ is an open covering family of X.

Question

Given

- objects E_i over U_i
- glueing data on $U_i \cap U_j$,
- satisfying the obvious cocycle condition on $U_i \cap U_j \cap U_k$

$\Rightarrow$ Does there exist E on X restricting to the E_i ?

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Given

- object $\widetilde{E}$ over L
- an action of $\mathrm{Gal}(L/K)$ on $\widetilde{E}$

$\Rightarrow$ Does there exist E over K such that $E \otimes_K L = \widetilde{E}$?

(My) Favorite descent example

Consider the algebraic group $\mathbb{G}_m$ over $\mathbb{C}$

We want to find actions of Gal$(\mathbb{C}/\mathbb{R}) = \langle \tau \rangle$ in order to obtain algebraic groups over $\mathbb{R}$.

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1. $\tau.x = \overline{x}$ $\rightsquigarrow$ taking invariants under this action yields the group $\mathbb{G}_m$ over $\mathbb{R}$
2. $\tau.x = (\overline{x})^{-1}$ $\rightsquigarrow$ here we obtain SO_2 over $\mathbb{R}$

Descent for Picard-Vessiot extensions

Theorem

For any algebraic Galois extension L/K there is an equivalence of categories

$$(\mathrm{PV}/K) \longrightarrow (\mathrm{PV}/L)^{\mathrm{Gal}(L/K)}$$

where

- (PV/K) *= category of Picard-Vessiot extensions with constants K*
- $(\mathrm{PV}/L)^{\Gamma}$ *= all objects are equipped with a Γ-action and the morphisms are Γ-equivariant*