Tutorial on Differential Galois Theory III

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02/14/08 / Oberflockenbach

Outline

Today's plan

- Monodromy and singularities
- Riemann-Hilbert correspondence and applications
- Irregular singularities: Stokes' approach
- Irregular singularities: Tannaka's approach
- Tannakian categories

Monodromy I

Consider the ∂-equation

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{z} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{with} \quad Y = \begin{pmatrix} 1 & \ln(z) \\ 0 & 1 \end{pmatrix}$$

as fundamental solution matrix.

- $\ln(z)$ is not defined globally on $\mathbb{C}^*$
- Analytic continuation: $\ln(z) \mapsto \ln(z) + 2\pi i$

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We obtain a representation

$$\pi_1(\mathbb{C}^*) \longrightarrow \mathrm{GL}_2(\mathbb{C}), \quad n \mapsto \begin{pmatrix} 1 & 2\pi i\, n \\ 0 & 1 \end{pmatrix}$$

Monodromy II

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- Equation $\frac{d}{dz}y = Ay$ with $A \in \mathbb{C}(z)^{n \times n}$
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Definition

The map

$$\mathcal{M}:\ \pi_1(\mathbb{P}^1\backslash S, p) \longrightarrow \mathrm{GL}_n(\mathbb{C}), \quad \gamma \mapsto C_\gamma$$

is called the monodromy representation.

Riemann-Hilbert problem

We obtain

$$\begin{pmatrix} \text{linear } \partial\text{-equations} \\ \text{with singularities} \\ \text{in } S \subset \mathbb{P}^1 \end{pmatrix} \xrightarrow{\mathcal{M}} \begin{pmatrix} \text{complex linear} \\ \text{representations} \\ \text{of } \pi_1(\mathbb{P}^1 \backslash S) \end{pmatrix}$$

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Riemann-Hilbert problem

This can't work:

We considered the error function as a solution to the equation $y'' + 2zy' = 0$.

- The equation is non-trivial (its solutions are non-elementary)
- Only one potential singularity at $\infty \Rightarrow$ trivial monodromy

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1. put $w = z^{-1}$ then $\frac{d}{dz} = -w^2 \frac{d}{dw}$

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$$\frac{d}{dw}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & (\frac{2}{w^3} - \frac{2}{w}) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$\Rightarrow$ The matrix defining the system has poles of order 3.

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Two systems $[A]$ and $[B]$ are called gauge-equivalent over F

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- A system $[A]$ is called regular singular at $x \in \mathbb{P}^1$ if it is gauge-equivalent over $\mathbb{C}(\{z - x\})$ to a system with at most simple poles.
- Otherwise we say $[A]$ is irregular singular at x.

Riemann-Hilbert correspondence (2nd attempt)

Theorem

The functor

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is an equivalence of categories.

Implications for differential Galois theory

Given a $A \in \mathbb{C}(z)^{n \times n}$, the group $\mathcal{M}(\pi_1(\mathbb{P}^1 \backslash S)) \subset \mathrm{GL}_n$ is called the monodromy group.

Theorem

Assume $A \in \mathbb{C}(z)^{n \times n}$ has only regular singularities. Then the differential Galois group of $[A]$ is isomorphic to the Zariski closure of the monodromy group.

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3. If $f \in E$ is invariant under monodromy then it extends to a holomorphic function on $\mathbb{P}^1 \backslash S$.
4. Use the fact that all solutions near a regular singular point have moderate growth $\mathrm{O}(|z|^{-N})$ to conclude that f is meromorphic.

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Given a linear algebraic group G over $\mathbb{C}$, does there exist a ∂-equation over $\mathbb{C}(z)$ which realizes G as its ∂-Galois group?

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Lemma

Every linear algebraic group G contains finitely many elements which generate G in the Zariski topology.

Proof. By induction on $\dim(G)$.

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4. Choose the unique representation of $\pi_1(\mathbb{P}^1 \backslash S)$ which maps γ_i to C_i.
5. Apply Riemann-Hilbert.

Irregular singularities? - Stokes' approach

Riemann-Hilbert classifies regular singular equations by monodromy data.

Classification of irregular singularities

- Recall: regularity of a singularity $\Leftrightarrow$ moderate growth of solutions

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- Let V_θ be the space of solutions on a small sector around θ
- We can decompose V_θ into a direct sum of subspaces defined by the various growth rates
- The obstruction for the existence of a global growth decomposition around the singularity is called the Stokes phenomenon

Irregular singularities? - Stokes' approach

Theorem (Deligne,Turittin,Hukuhara,Malgrange)

There is a functor

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which is an equivalence.

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Again we can use the classification data to compute the differential Galois group.

Theorem (Ramis)

The differential Galois group of an equation over $\mathbb{C}(z)$ is topologically generated by the monodromy group and the Stokes matrices.

Irregular singularities? - Tannaka's approach

The failure of Riemann-Hilbert for irregular singularities can be regarded as a motivation for the Tannakian approach.

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\end{array}$$

In this sense, the group π_1^{diff} can be understood as an enhancement of the topological fundamental group π_1^{top}. It is called the Tannakian fundamental group.

Tannakian categories

Definition

A Tannakian category over K is a rigid abelian tensor category $\mathcal{C}$ over K with an exact faithful K-linear functor

$$\omega : \mathcal{C} \longrightarrow (\text{Vect}/K)$$

called the fiber functor.

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Theorem (Deligne, Grothendieck, Milne, Saavedra)

Let $\mathcal{C}$ be a Tannakian category over K. Then

- $\mathrm{Aut}^{\otimes}(\omega)$ *is isomorphic to an affine group scheme G over K.*
- *there is an equivalence of abelian tensor categories*

$$\mathcal{C} \xrightarrow[\cong]{\omega} (\mathrm{Rep}(G)/K)$$

Examples of Tannakian categories (I)

$\mathbb{Z}$-graded vector spaces

- Let V complex representation of the group $\mathbb{G}_m, \mathbb{C}$
- V admits a weight space decomposition

$$V = \bigoplus_{\chi \in \mathrm{char}(\mathbb{G}_m)} V_\chi$$

- $\mathrm{char}(\mathbb{G}_m) = \mathbb{Z}$

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⇒ The category of $\mathbb{Z}$-graded complex vector spaces is Tannakian over $\mathbb{C}$ with fundamental group $\mathbb{G}_m$.

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Mixed real Hodge structures

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- Consider the group $G = \mathbb{C}^*/\mathbb{R}$ as a real algebraic group
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- Let V be a real representation of G
- $V \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{(p,q) \in \mathbb{Z} \oplus \mathbb{Z}} V^{p,q}$
- $\overline{V^{p,q}} = V^{q,p}$

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$\Rightarrow$ The category of mixed real Hodge structures is Tannakian over $\mathbb{R}$ with fundamental group $\mathbb{C}^*/\mathbb{R}$.

$\mathcal{D}$-modules

Definition

Let F be a ∂-field.

- *Define the non-commutative ring*

$$\mathcal{D} = F[\partial] \quad \textit{subject to} \quad \partial f = f\partial + \partial(f)$$

  of differential operators over F.
- *A $\mathcal{D}$-module is module over $\mathcal{D}$, finite dimensional over F.*

$\mathcal{D}$-modules vs. ∂-equations

- Let M be a $\mathcal{D}$-module.
- Let $\underline{e} = (e_1, \ldots, e_n)$ be an F-basis.
- There exists $A \in F^{n \times n}$ such $\partial(\underline{e}) = \underline{e}A$

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- Assume $\underline{f} = (f_1, \ldots, f_n)$ is another basis with $\partial(\underline{f}) = \underline{f}B$
- then there exists $C \in \mathrm{GL}_n(F)$ such that $\underline{e} = \underline{f}C$

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- then there exists $C \in \mathrm{GL}_n(F)$ such that $\underline{e} = \underline{f}C$
- So

$$\begin{aligned}\underline{e}A &= \partial(\underline{e}) = \partial(\underline{f}C) \\ &= \partial(\underline{f})C + \underline{f}\partial(C) \\ &= \underline{e}(C^{-1}BC + C^{-1}\partial(C))\end{aligned}$$

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- Let M be a $\mathcal{D}$-module.
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- There exists $A \in F^{n \times n}$ such $\partial(\underline{e}) = \underline{e}A$
- Assume $\underline{f} = (f_1, \ldots, f_n)$ is another basis with $\partial(\underline{f}) = \underline{f}B$
- then there exists $C \in \mathrm{GL}_n(F)$ such that $\underline{e} = \underline{f}C$
- So

$$\begin{aligned} \underline{e}A &= \partial(\underline{e}) = \partial(\underline{f}C) \\ &= \partial(\underline{f})C + \underline{f}\partial(C) \\ &= \underline{e}(C^{-1}BC + C^{-1}\partial(C)) \end{aligned}$$

$\Rightarrow \; A = C^{-1}BC + C^{-1}\partial(C)$

Conclusion

A $\mathcal{D}$-module is an intrinsic description of a gauge-equivalence class of ∂-equations.

The ultimate example of a Tannakian category

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The Tannakian fundamental group of $\langle M \rangle^{\otimes}$ is isomorphic to the differential Galois group of any equation in the gauge equivalence class given by M.

A reality check with Tannaka

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Assume $A \in \mathbb{C}(z)^{n\times n}$ with $\mathrm{trace}(A) = 0$. Then the ∂-Galois group G of A is isomorphic to a subgroup of $\mathrm{SL}_n(\mathbb{C})$.

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