MATH 114 - EXTRA CREDIT PROBLEM # 7

Abstract. You can turn in this problem anytime before the final exam. Work on extra credit problems will be used to decide cases in which a course grade is in between two letter grades.

Balancing blocks

Suppose that we take n-identical rectangular blocks which have length 1, height h and width d. Begin with the blocks stacked directly on top of each other on the $x-y$ plane so that they straddle the negative x-axis, with their leading edges over $x=0$ and their trailing edges over $x=-1$. The j^{th} block then lies between the planes $z=(n-j)h$ and $z=(n-j+1)h$.

Slide the top block (which we'll call block #1) a distance 1/2 in the positive x direction along the block underneath it (block #2). Then slide block #2 a distance 1/4 in the positive x-direction along the block underneath it (block #3) while letting block #1 ride along on top of block #2. Continue doing this for all the blocks by sliding the j^{th} block (along with all the blocks on top of it) a distance $1/(2j)$ along the block underneath it (the $(j+1)^{st}$ block) for each integer $1 \le j \le n$. In the case of the n^{th} block, this means that we slide the n^{th} block and all blocks on top of it a distance $1/(2n)$ in the positive x-direction along the x-axis. (Think of the $x-y$-plane as the table on which the blocks sit.)

The goal of this exercise is to find the x coordinate $\overline{x}$ of the center of mass of the blocks assuming that each one has constant density function equal to 1 within the block.

(1) Call the mid-point of each block the point which is exactly in the center of the block. Show that x-coordinate of the mid-point of the j^{th} block is

$$x_j = (1/2 + 1/4 + 1/6 + \cdots + 1/(2j)) - 1/2.$$

Hint: Where is the right leading edge of the j^{th} block?

(2) Show that the density function $\rho_j(x,y,z)$ associated to the j^{th} block has the following properties:

$$\rho_j(x,y,z) = \begin{pmatrix} 1 & \text{if} \quad x_j - 1/2 \le x \le x_j + 1/2, \quad |y| \le d/2 \quad \text{and} \quad (n-j)h \le z \le (n-j+1)h \\ 0 & \text{otherwise} \end{pmatrix}$$

(3) Show that the x-coordinate of the center of mass of the j^{th} block is x_j, as one would expect. (Hints: The mass of the j-block is $M(j) = \int\int\int \rho_j(x,y,z)dV$. Show this is the volume $1 \cdot d \cdot h = dh$ of the block. Then calculate the moment

$$M(j)_{yz} = \int\int\int x\rho_j(x,y,z)dV$$

using the formula in problem #2. Show $x_j = M(j)_{yz}/M(j)$.)

(4) Using the work in problem #3, show that the x coordinate $\overline{x}$ of all the totality of all the blocks is

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_{2n}}{n}.$$

(Hints: The mass of all the blocks is the sum of their individual masses. Show that the moment with respect to the yz-plane is the sum of the corresponding moments for each block. Then use that all the blocks have the same mass.)

(5) Show that $\overline{x} = 0$. Deduce from this that the stack of blocks will balance if they are put on a table whose edge is the line $x = 0$ in the x-y plane.

Hints: Write

$$\begin{array}{rcccccccc} x_1 + 1/2 & = & 1/2 & + & 1/4 & + & 1/6 & + & \cdots \\ x_2 + 1/2 & = & & & 1/4 & + & 1/6 & + & \cdots \\ x_3 + 1/2 & = & & & & & 1/6 & + & \cdots \\ \cdots & & & & & & & & \cdots \end{array}$$

Add up both sides and show $x_1 + x_2 + \cdots + x_n = 0$.

(6) Euler's constant is defined to be

$$\gamma = \lim_{n\to\infty} \left(\left(\sum_{j=1}^{j=n} \frac{1}{j} \right) - \ln(n) \right) = 0.5771\ldots$$

Explain why the x coordinate of the leading edge of the top block equals

$$\ln(n) + \gamma = \ln(n) + 0.5771\ldots$$

plus a function which goes to 0 as $n \to +\infty$. Suppose the length of the block is 1 foot, and the height h of each block is 1-inch. How many miles high will the stack of blocks have to be in order to put the leading edge of the top block 20 feet from the edge of the table? One light year is 5.878×10^{12} miles. If the stack of blocks is one light year high, how many feet will the far end of the top block be from the edge of the table?