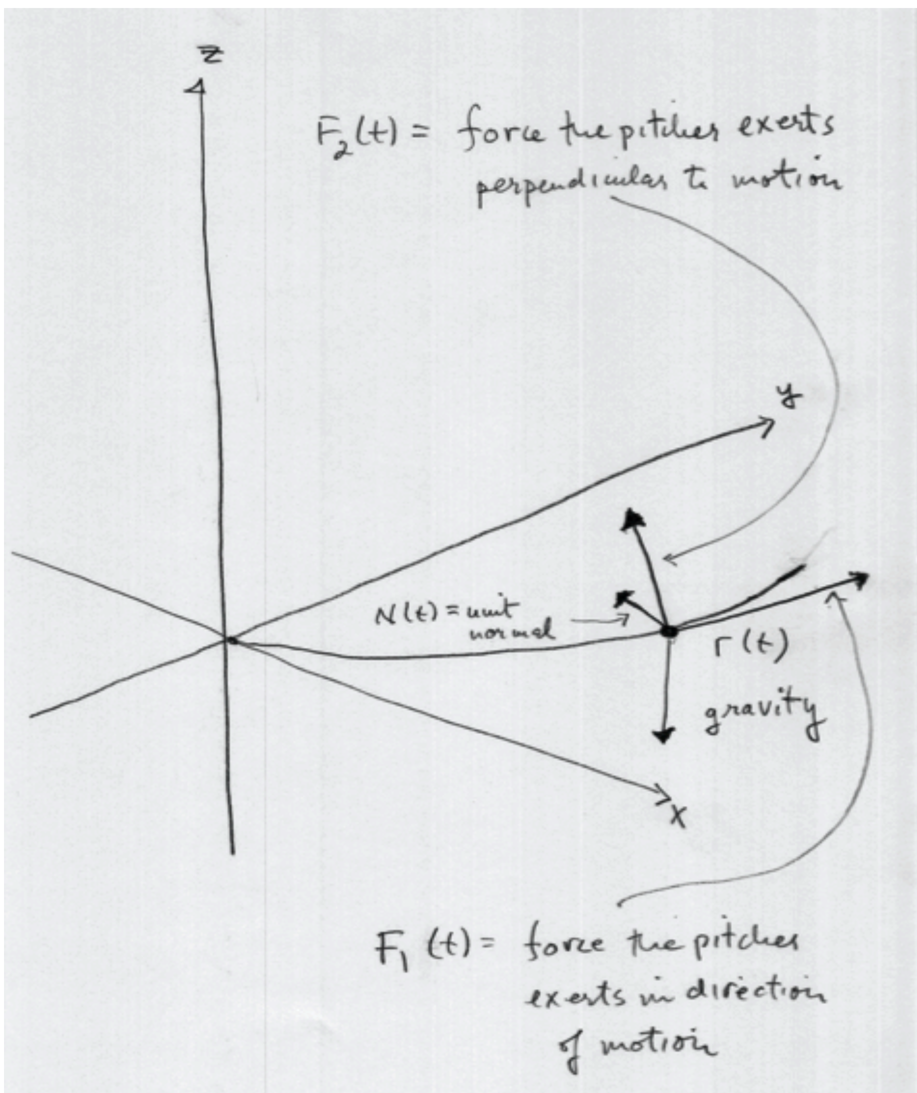
BASEBALL PITCHES AND COMPONENTS OF ACCELERATION

Statement of the Problem: A baseball pitcher throws a baseball of mass m. The force on the baseball is the sum of the force exerted by the pitcher together with a constant downward gravitational force equal to $(0, 0, -mg)$. The ball follows the path described by

$$r(t) = (t, t^2/2, 0) \quad \text{for} \quad 0 \leq t \leq 1$$

where t is time. The pitcher can exert a force of magnitude at most f in a direction perpendicular to the motion of the ball before he hurts his arm. Find a formula which determines from m, g and f whether this is a safe pitch for the pitcher. If the pitch is unsafe, will an injury occur at the beginning of the pitch (when $t = 0$) or near the end (when t is close to 1)?



To solve this problem, let $F_P(t)$ be the total force vector the pitcher exerts on the ball. We can write

$$F_P(t) = F_1(t) + F_2(t) \tag{0.1}$$

with $F_1(t)$ the force in the direction of motion of the ball at time t and $F_2(t)$ the component of $F(t)$ which is perpendicular to the direction of motion. The total force on the ball is then

$$F(t) = F_P(t) + (0, 0, -mg) \tag{0.2}$$

since gravity also acts on the ball.

By Newton's law,

$$F(t) = mr''(t) \tag{0.3}$$

Write

$$r'(t) = (1, t, 0) \quad \text{and} \quad s'(t) = |r'(t)| = (1 + t^2)^{1/2} \tag{0.4}$$

Let $T(t)$ and $N(t)$ be the unit tangent vector and the unit normal vector at time t. We could calculate these, but in fact we don't need to know them explicitly.

The curvature $\kappa(t)$ of the path traced by $r(t)$ can be found most easily using the fact that

$$r(t) = (t, t^2/2, 0)$$

is the curve in the $x - y$ plane which is the graph of the function $f(x) = x^2/2$. Here $f'(x) = x$ and $f''(x) = 1$. We discussed in class that the curvature of such graphs is given by

$$\kappa(t) = \frac{|f''(t)|}{(1 + f'(t)^2)^{3/2}} = \frac{1}{(1 + t^2)^{3/2}} = (1 + t^2)^{-3/2} \tag{0.5}$$

We can now use the decomposition of acceleration into tangential and normal components. From (0.4) and (0.5) the formula for this discussed in class becomes:

$$\begin{aligned} r''(t) &= s''(t)T(t) + \kappa(t)s'(t)^2 N(t) \\ &= s''(t)T(t) + (1 + t^2)^{-3/2}(1 + t^2)N(t) \\ &= s''(t)T(t) + (1 + t^2)^{-1/2} N(t) \end{aligned} \tag{0.6}$$

From Newton's Law (0.3), (0.6), (0.2) and (0.1) we get

$$\begin{aligned} F(t) &= mr''(t) \\ &= ms''(t)T(t) + m(1 + t^2)^{-1/2} N(t) \\ &= F_1(t) + F_2(t) + (0, 0, -mg) \end{aligned} \tag{0.7}$$

Here $F_1(t)$ is the force the pitcher exerts in the direction of motion, and this is a multiple of the unit tangent vector $T(t) = (1, t, 0)/\sqrt{1 + t^2}$. The vector $N(t)$ is perpendicular to $T(t)$. The vector $F_2(t)$ is perpendicular to $T(t)$ since this is the force the pitcher exerts perpendicular to the direction of motion. The vector $(0, 0, -mg)$ is also perpendicular to $T(t)$, as one can see by taking a dot product or by simply from the fact that gravity acts in the negative z direction, which is perpendicular to any vector in the $x - y$ plane.

We now equate the components of $F(t)$ in the last two equations of (0.7) which are perpendicular to $T(t)$. This gives

$$m(1 + t^2)^{-1/2} N(t) = F_2(t) + (0, 0, -mg) \tag{0.8}$$

Thus $F_2(t)$ is

$$F_2(t) = m(1 + t^2)^{-1} N(t) + (0, 0, mg) \tag{0.9}$$

We are interested in the magnitude of $F_2(t)$. The vector $T(t) = r'(t)/|r'(t)|$ has z-component equal to 0. On taking derivatives of the components of $T(t)$, we see that $T'(t)$ also has z-component equal to 0. So $N(t) = T'(t)/|T'(t)|$ also has z-component equal to 0. Therefore $N(t)$ lies in the

x-y-plane, so $N(t)$ is perpendicular to $(0, 0, mg)$. This means that we can apply the pythagorean theorem to the find length of $F_2(t)$ in (0.9). This gives

$$\begin{aligned} |F_2(t)|^2 &= |m(1+t^2)^{-1}N(t)|^2 + |(0,0,mg)|^2 \\ &= m^2(1+t^2)^{-2}|N(t)|^2 + (mg)^2 \\ &= m^2((1+t^2)^{-2} + g^2) \end{aligned} \tag{0.10}$$

where we have $|N(t)| = 1$ since $N(t)$ is a unit vector.

Taking square roots in (0.10) shows

$$|F_2(t)| = m\sqrt{(1+t^2)^{-2} + g^2} \tag{0.11}$$

This will be largest for $t = 0$, so

$$\max\{|F_2(t)| : 0 \leq t \leq 1\} = |F_2(0)| = m\sqrt{1+g^2} \tag{0.12}$$

So the pitch will be safe provided

$$m\sqrt{1+g^2} \leq f = \text{maximum safe force perpendicular to motion} \tag{0.13}$$

If the pitch is not safe, (0.12) shows that $|F_2(0)|$ will exceed f. So the most dangerous moment for the pitcher is in fact $t = 0$, right when the pitch starts.