

## Power series problems.

These notes give some advice about how to approach power series problems as well as detailed solutions to problems of this kind which appear in the old final exams at the end of the math 114 lab manual.

### 1. GENERAL ADVICE

1. Power series problems tend to involve more computation and be more time consuming than other problems. Thus you would be wise to leave them until after you have looked over the exam for simpler problems.
2. You may be able to find an explicit solution to a given differential equation and then the power series for this solution. This is the case for the problem discussed in the next section, for example. Keep in mind the following power series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

3. If you cannot see a way to find a power series for an explicit solution, the method we discussed in class was to write a potential solution as  $y(x) = \sum_{n=0}^{\infty} a_n x^n$ . One then has

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

Substitute these expressions into the differential equation. You will generally arrive at a sum of several sums, each of which has the form

$$\sum_{n=N}^{\infty} f(n)x^{n+b}$$

where  $f(n)$  can depend on several of the  $a_n$ 's and  $b$  is an integer. Different sums can have different values for  $N$  and for  $b$ . You want to group together all the coefficients of like powers of  $x$  in all of these sums. To do this, change variables in each one separately by letting  $m = n + b$ , so that  $n = m - b$ . Changing variables from  $n$  to  $m$  leads to

$$\sum_{n=N}^{\infty} f(n)x^{n+b} = \sum_{m=n+b=N+b}^{\infty} f(m-b)x^m$$

If you do this for each sum, you will be able to group together the coefficients of  $x^m$  once  $m$  is sufficiently large. Small values of  $m$  may appear in some of the sums and not in others, so you need to deal with these separately. In the end the differential equation will be equivalent to an expression of the form

$$\sum_m c(m)x^m = 0$$

in which  $m$  runs over some range and  $c(m)$  depends on the original  $a_n$  coefficients in  $y(x) = \sum_{n=0}^{\infty} a_n x^n$ . This equation will hold if and only if  $c(m) = 0$  for all  $m$ . Use this to find how the  $a_n$  must depend on the first few of  $a_0, a_1, \dots$ . For some examples of this method, see the next sections.

## 2. PROBLEM # 9, MULTIPLE CHOICE SECTION, SPRING 2004

**Problem 9.** Suppose  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  is a solution of the differential equation

$$(1) \quad y'(x) + x^4 y(x) = 0.$$

What is the first value of  $n$  beyond  $n = 0$  for which the coefficient  $a_n$  can be non-zero?

**First Solution:** In this case, we can solve (1) explicitly using separation of variables. Write (1) as

$$\frac{dy}{dx} = -x^4 y.$$

Following the usual separation of variables approach, this is the same as

$$\frac{dy}{y} = -x^4 dx.$$

Integrating gives

$$\ln |y(x)| = -\frac{x^5}{5} + c$$

for some constant  $c$ . Exponentiating and using the power series for the exponential function shows

$$|y(x)| = e^{-x^5/5} e^c = e^c \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{-x^5}{5} \right)^m$$

where  $e^c$  is a constant. Notice that  $|y(x)|$  is never 0, so  $y(x)$  can never be 0. This means that  $y(x)$  can't change sign, since it is continuous. So there is choice of sign  $\pm$  such that  $y(x) = \pm |y(x)|$  for all  $x$ . We get

$$(2) \quad \sum_{n=0}^{\infty} a_n x^n = y(x) = \pm |y(x)| = \pm e^c \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{-x^5}{5} \right)^m = \sum_{m=0}^{\infty} \frac{\pm e^c (-1)^m}{5^m m!} x^{5m}.$$

This means  $a_0 = \pm e^c$  and

$$(3) \quad a_n = \frac{\pm e^c (-1)^m}{5^m m!} = \frac{a_0 (-1)^m}{5^m m!} \quad \text{if } n = 5m \quad \text{for some } m \quad \text{and } a_n = 0 \quad \text{otherwise.}$$

The smallest  $n > 0$  for which  $a_n$  can be non-zero is  $n = 5$ .

**Second Solution.** Finding an explicit solution is usually quicker, but it may not always be possible, and it may be difficult to find a power series for the solution. In this case, you will need to iteratively find the equations which the  $a_n$  have to satisfy.

Differentiating  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  term by term shows that

$$(4) \quad y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}.$$

Notice that the sum starts at  $n = 1$ , because the  $n = 0$  term in  $y(x)$  is the constant  $a_0$  which has derivative 0.

Plug (4) into the equation (1) which we want to solve. This shows

$$(5) \quad \sum_{n=1}^{\infty} a_n n x^{n-1} + x^4 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Since  $x^4$  multiplies each term in the second sum, and  $x^4 x^n = x^{n+4}$ , we can rewrite this as:

$$(6) \quad \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+4} = 0$$

The problem now is to rewrite this in such a way as to group together like powers of  $x$ . In each sum,  $x$  appears to a certain power. For instance, in the first sum,  $x$  appears to the power  $x^{n-1}$  when  $n$  is the running variable. In the second sum,  $x$  appears to the power  $n + 4$  when  $n$  is the running variable.

I recommend looking at the exponent to which  $x$  appears in each sum and then renaming this the new variable  $m$ . This  $m$  will be a different function of  $n$  for each sum. For instance, in the first sum in (6), one would want to use  $m = n - 1$ , and in the second sum one would want to use  $m = n + 4$ . For each sum separately, figure out what  $n$  is in terms of  $m$ , and rewrite that sum in terms of  $m$ . You will need to be careful to figure out what range  $m$  should run over to correspond to the original range over which  $n$  runs.

Let's do this first for the first sum  $\sum_{n=1}^{\infty} a_n n x^{n-1}$  which appears in (6). Here  $x$  appears to the power  $n - 1$ . So we let  $m = n - 1$ . This forces  $n = m + 1$ . Rewriting each term using  $m$  rather than  $n$  now shows

$$(7) \quad \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{m=n-1=0}^{\infty} a_{m+1} (m+1) x^m$$

Now look at the second sum

$$(8) \quad \sum_{n=0}^{\infty} a_n x^{n+4}$$

which appears in (6). Here  $x$  appears to the power  $n + 4$ . So we want to use  $m = n + 4$ , and  $n = m - 4$ . We now systematically change every appearance of  $n$  in (8) to  $m - 4$ . This leads to

$$(9) \quad \sum_{n=0}^{\infty} a_n x^{n+4} = \sum_{m=n+4=4}^{\infty} a_{m-4} x^m$$

Substituting (7) and (9) into (6) gives:

$$(10) \quad \sum_{m=0}^{\infty} a_{m+1} (m+1) x^m + \sum_{m=4}^{\infty} a_{m-4} x^m = 0$$

The power to which  $x$  occurs in the terms of each sum is the same, namely  $m$ , so we can combine terms. But notice that the ranges of the two sums are different. The first sum includes terms for  $m = 0, 1, 2, 3$  and the second sum does not. So when we take this into account, we end up with the formula

$$(11) \quad \sum_{m=0}^3 a_{m+1} (x+1) x^m + \sum_{m=4}^{\infty} (a_{m+1} (m+1) + a_{m-4}) x^m = 0$$

We want to determine all choices of the coefficients  $\{a_m\}_{m=0}^{\infty}$  for which (11) is true, since this equation is equivalent to the original differential equation. The equality (11) will be true if and only if the coefficient of  $x^m$  for each  $m$  is 0. This is the same as

$$(12) \quad a_{m+1} = 0 \quad \text{for } m = 0, 1, 2, 3 \quad \text{and} \quad a_{m+1} (m+1) + a_{m-4} = 0 \quad \text{for } m \geq 4.$$

We can rewrite this as:

$$(13) \quad a_m = 0 \quad \text{for } m = 1, 2, 3, 4 \quad \text{and} \quad a_{m+1} = -a_{m-4}/(m+1) \quad \text{for } m \geq 4.$$

The problem asked for the least  $m > 0$  for which  $a_m$  can be non-zero. Notice that the first condition in (13) certainly forces  $a_m = 0$  for  $m = 1, 2, 3, 4$ . The second condition then forces  $a_5 = -a_0/5$ . In fact, we can rewrite the second part of (13) by changing variables again to  $n = m + 1$ . The condition becomes

$$(14) \quad a_m = 0 \quad \text{for } m = 1, 2, 3, 4 \quad \text{and} \quad a_n = -a_{n-5}/n \quad \text{for } n \geq 5.$$

So for  $n \geq 5$  we get

$$(15) \quad a_n = \frac{-a_{n-5}}{n} = \frac{a_{n-10}}{n(n-5)} = \dots = \frac{(-1)^t a_j}{n(n-5)(n-10) \cdots (j+5)}$$

where  $j \in \{0, 1, 2, 3, 4\}$  is the integer for which  $n = j + 5t$  for some integer  $t$ . Conversely, if (15) holds and  $a_m = 0$  for  $m = 1, 2, 3, 4$  then (13) is true and the original differential equation will hold.

We conclude from this that  $a_0$  determines all the  $a_i$ , and that  $n = 5$  is the smallest positive index for which  $a_n$  can be non-zero. This agrees with the quick and dirty solution we found by separation of variables. In fact, if you look at (15) and (13) closely, you see that  $a_j = 0$  unless  $j = 0$ , so  $a_n = 0$  unless  $n = 5t$  is a multiple of 5. Then (15) is the same as

$$a_{5t} = \frac{(-1)^t a_0}{5t(5t-5)(5t-10) \cdots 5} = \frac{(-1)^t a_0}{5^t t!}$$

This agrees with the equality (3).

## 3. PROBLEM 2, FREE RESPONSE SECTION, SPRING 2004 EXAM

**Problem:** Solve  $y''(x) - xy'(x) + 2y(x) = 0$  subject to  $y(0) = 1$  and  $y'(0) = 1$  by power series. Write out all non-zero terms in the solution through the term in  $x^5$ .

**Solution:** We don't have a technique for finding an explicit solution, so we solve for the conditions on the coefficients  $a_n$  in a power series expansion  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  which are equivalent to  $y(x)$  being a solution of the differential equation.

We have

$$(16) \quad y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$(17) \quad y''(x) = \sum_{n=2}^{\infty} a_n n (n-1) x^{n-2}.$$

Substituting into the differential equation shows this is the same as

$$(18) \quad \sum_{n=2}^{\infty} a_n n (n-1) x^{n-2} - x \sum_{n=1}^{\infty} a_n n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Multiplying the second and third sums by the terms just to the left of them gives

$$(19) \quad \sum_{n=2}^{\infty} a_n n (n-1) x^{n-2} - \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0.$$

We change variables to  $m = n - 2$  in the first sum and to  $m = n$  in the second and third. This leads to

$$(20) \quad \sum_{m=n-2=0 \text{ so } n=m+2}^{\infty} a_{m+2} (m+2) (m+1) x^m - \sum_{m=1}^{\infty} a_m m x^m + \sum_{m=0}^{\infty} 2 a_m x^m = 0.$$

The only obstacle to combining these three sums is that the middle sum starts at  $m = 1$  while the first and third start at  $m = 0$ . A foolproof way to deal with this is to separate out the terms of the first and third sum which correspond to  $m = 0$ , and to group together the terms of all three sums starting at  $m = 1$ . This leads to

$$(21) \quad a_2 \cdot 2 \cdot 1 + 2a_0 + \sum_{m=1}^{\infty} (a_{m+2} (m+2) (m+1) - a_m m + 2 a_m) x^m = 0.$$

The coefficient of  $x^m$  has to be 0 for every  $m$ . So this equality is the same as

$$(22) \quad 2a_2 + 2a_0 = 0 \quad \text{and} \quad a_{m+2} (m+2) (m+1) - a_m m + 2 a_m = 0 \quad \text{for } m \geq 1.$$

Notice that the first equality here is what the second one gives if we set  $m = 0$ . Furthermore,

$$a_{m+2} (m+2) (m+1) - a_m m + 2 a_m = 0$$

is the same as

$$a_{m+2} (m+2) (m+1) = (m-2) a_m.$$

So we can simplify (22) to

$$(23) \quad a_{m+2} = \frac{(m-2)a_m}{(m+2)(m+1)} \quad \text{for } m \geq 0.$$

Since  $y(x) = \sum_{m=0}^{\infty} a_m x^m$ , we have  $1 = y(0) = a_0$  and  $1 = y'(0) = a_1$ . Setting  $m = 0$  in (23) gives

$$a_2 = \frac{(0-2)a_0}{(0+2)(0+1)} = \frac{-2a_0}{2} = -1.$$

Setting  $m = 1$  in formula (23) gives

$$a_3 = \frac{(1-2)a_1}{(1+2)(1+1)} = \frac{-a_1}{3 \cdot 2} = \frac{-1}{6}$$

Setting  $m = 2$  in (23) gives

$$a_4 = \frac{(2-2)a_2}{(2+2)(2+1)} = 0.$$

Setting  $m = 3$  in (23) gives

$$a_5 = \frac{(3-2)a_3}{(3+2)(3+1)} = \frac{a_3}{20} = \frac{-1}{120}.$$

This is the answer given in the answer key in the lab manual, and solves the stated problem.

It's still interesting to try to determine all of the coefficients  $a_m$ . Notice that (23) implies that  $a_6$  is a rational multiple of  $a_4 = 0$ , so  $a_6 = 0$ . Similarly,  $a_8$  is a rational multiple of  $a_6$ , so  $a_8 = 0$  and in fact  $a_m = 0$  if  $m \geq 4$  is even. If  $m = 2t + 1$  is odd, then (23) gives

$$\begin{aligned} a_{m+2} &= \frac{(m-2)a_m}{(m+2)(m+1)} \\ &= \frac{(m-2)(m-4)a_{m-2}}{(m+2)(m+1)m(m-1)} \\ &= \dots \\ &= \frac{(m-2)(m-4)\cdots 1 \cdot (-1) \cdot a_1}{(m+2)!} \\ &= -\frac{(m-2)!a_1}{(m-3)(m-5)(m-7)\cdots(2) \cdot (m+2)!} \\ &= -\frac{(m-2)!a_1}{(2t-2)(2t-4)\cdots(2) \cdot (m+2)!} \\ &= -\frac{(m-2)!a_1}{2^{t-1}(t-1)!(m+2)!} \end{aligned}$$

Now if  $m \geq 3$  is odd, then

$$\frac{(m-2)!}{(m+2)!} = \frac{1}{(m+2)(m+1)m(m-1)}$$

and  $t = (m - 1)/2$  when  $m = 2t + 1$ . So (24) shows that if  $m \geq 3$  is odd then

$$\begin{aligned} a_{m+2} &= -\frac{a_1}{2^{(m-3)/2}((m-3)/2)!(m+2)(m+1)m(m-1)} \\ &= \frac{-1}{2^{(m-3)/2}((m-3)/2)!(m+2)(m+1)m(m-1)} \end{aligned}$$

since  $a_1 = 1$ . As a check, setting  $m = 3$  gives

$$a_5 = \frac{-1}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{-1}{120}$$

as before.