Math 114-001: Hour Exam 2 Key

Feb. 22, 2008

Multiple choice questions (10 points each).

(1) The curves in $\mathbb{R}^3$ defined by $r(t) = (t^2, \sin(t), t^3)$ and $h(t) = (\sin(t), t, t^2)$ intersect at the point $(0,0,0)$ when $t = 0$. What is the angle in radians between these two curves at the point $(0,0,0)$?

(A) 0 (B) $\pi/2$
(C) $\pi/3$ (D) $\pi/4$
(E) $2\pi/3$ (F) none of the above

Answer to 1: D

The tangent lines to curves have direction vectors $r'(0)$ and $h'(0)$, where $r'(t) = (2t, \cos(t), 3t^2)$ and $h'(t) = (\cos(t), 1, 2t)$. Thus the angle θ between the two curves is the angle between $r'(0) = (0,1,0)$ and $h'(0) = (1,1,0)$. Thus

$$\cos(\theta) = \frac{r'(0) \cdot h'(0)}{|r'(0)||h'(0)|} = \frac{(0,1,0) \cdot (1,1,0)}{1 \cdot \sqrt{2}} = \frac{1}{\sqrt{2}}.$$

This means $\theta = \pi/4$.

(2) Two snails are at positions $r(t) = (t^2, 2 - t^2)$ and $d(t) = (t, t)$ in the x-y plane at time t. What are all the values of t for which the snails are at the same position?

(A) $t = 1$ and $t = -1$ (B) $t = 1$ and $t = 0$
(C) $t = 1$ (D) $t = -1$
(E) no values of t (F) none of the above.

Answer to 2: C

We want to know all values of t for which $t^2 = t$ and $2 - t^2 = t$.
The first equation $t^2 = t$ has solutions $t = 0$ and $t = 1$.
The equation $2 - t^2 = t$ is not satisfied by $t = 0$, but it is satisfied by $t = 1$.
Thus $t = 1$ is the only common solution.

(3) Find the curvature function $\kappa(t)$ of the parametric curve described by $r(t) = (1, \cos(t), \sin(t))$.

(A) $\kappa(t) = \sin(t)$

(B) $\kappa(t) = \cos(t)$

(C) $\kappa(t) = -1$

(D) $\kappa(t) = 1$

(E) none of the above

Answer to 3: D

Use the formula $\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$. One has $r'(t) = (0, -\sin(t), \cos(t))$, $r''(t) = (0, -\cos(t), -\sin(t))$, $|r'(t)| = 1$, $r'(t) \times r''(t) = (1, 0, 0)$ so $\kappa(t) = 1$. This fits with the fact that the curve parameterized by $r(t)$ is a circle of radius 1 parallel to the y-z plane.

(4) Let C be the curve parameterized by $r(t) = (t, t^2, t^3)$. An equation in rectangular coordinates (x, y, z) for the normal plane to C at the point $r(1) = (1, 1, 1)$ is:

(A) $x - 1 = 0$

(B) $(x - 1) + 3(y - 1) + 2(y - 1) = 0$

(C) $(x - 1) + 2(y - 1) + 3(z - 1) = 0$

(D) $x + y + z = 3$

(E) none of the above

Answer to 4: C

The normal plane is the one containing $r(1) = (1, 1, 1)$ which is perpendicular to the tangent direction $r'(1)$. Here $r'(t) = (1, 2t, 3t^2)$ so $r'(1) = (1, 2, 3)$. Thus the tangent plane is the set of points (x, y, z) such that

$$0 = ((x, y, z) - (1, 1, 1)) \cdot (1, 2, 3) = (x - 1) + 2(y - 1) + 3(z - 1).$$

(5) Let C be the curve parameterized by the function $r(t) = (t, t^2, t^3)$. Which of the following is an equation for the tangent line to C at the point $r(1) = (1, 1, 1)$?

(A) $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$

(B) $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1}$

(C) $\frac{x-1}{3} = \frac{y-1}{2} = \frac{z-1}{1}$

(D) $x - 1 = 2(y-1) = 3(z-1)$

(E) none of above

Answer to 5: A

The derivative of $r(t)$ is $r'(t) = (1, 2t, 3t^2)$, so a direction vector for tangent line at $r(1) = (1, 1, 1)$ is $r'(1) = (1, 2, 3)$. This means the the line is the set of all points $(1, 1, 1) + u(1, 2, 3)$ for $u \in \mathbb{R}$, which leads to answer (A).

(6) A tennis player upset at an umpire's call launches their racket toward the umpire's viewing stand. When it leaves the player's hand, the racket is at a height of 2 feet. It is launched toward the viewing stand at an angle of 45 degrees from horizontal and at a speed of 20 feet per second. The downward acceleration of gravity is $g = 32\frac{\text{ft}}{\text{sec}^2}$. At what height will the racket hit the viewing stand if the viewing stand is 10 feet away? You should neglect air resistance and the possibility that the racket will hit someone en route.

(A) 1 foot

(B) 2 feet

(C) 3 feet

(D) $2\frac{1}{2}$ feet

(E) 4 feet

(F) none of the above.

Answer to 6: E

Let $r(t) = (x(t), y(t))$ be the position of the racket at time t. We have

$$r(0) = (0, 2)$$

$$r'(0) = 20 \cdot (\cos(45°), \sin(45°)) = 20(1/\sqrt{2}, 1/\sqrt{2}) = 10\sqrt{2}(1, 1)$$

and

$$r''(t) = (0, -g) = (0, -32).$$

Then

$$
\begin{aligned}
r'(T) &= r'(0) + \int_{t=0}^{t=T} r''(t)dt \\
&= 10\sqrt{2}(1,1) + \int_{t=0}^{t=T} (0,-32)dt \\
&= 10\sqrt{2}(1,1) + (0,-32T) \\
&= (10\sqrt{2}, 10\sqrt{2} - 32T)
\end{aligned}
$$

and

$$
\begin{aligned}
r(T) &= r(0) + \int_{t=0}^{t=T} r'(t)dt \\
&= (0,2) + \int_{t=0}^{t=T} (10\sqrt{2}, 10\sqrt{2} - 32T)dt \\
&= (10\sqrt{2}T, 2 + 10\sqrt{2}T - 16T^2) = (x(T), y(T)).
\end{aligned}
$$

The time T_0 when the racket hits the stand is when

$$
x(T_0) = 10\sqrt{2}T_0 = 10 \quad \text{so} \quad T_0 = \frac{10}{10\sqrt{2}} = \frac{1}{\sqrt{2}}.
$$

The height of the racket at T_0 is

$$
y(T_0) = 2 + 10\sqrt{2}T_0 - 16T_0^2 = 2 + 10 - 16/2 = 12 - 8 = 4.
$$

(7) Suppose a particle has position $r(t) = (1, -1, t^2)$ at time $t > 0$. Find the magnitude of the component of the acceleration of the particle at time t which is perpendicular to the direction of motion of the particle.

(A) $|2t|$ (B) 2 (C) 0

(D) 1 (E) $|t|$ (F) none of the above

Answer to 7:

The motion of the particle is in the direction of $r'(t) = (0, 0, 2t)$, and the acceleration is $r''(t) = (0, 0, 2)$. Since $r''(t)$ is a scalar multiple of $r'(t)$, the component of acceleration which is perpendicular to $r'(t)$ has magnitude 0.

(8) Suppose that a curve C is parameterized by $r(t) = (\cos(t), \sin(t), \sqrt{8} \cdot t)$. Let $s = s(t)$ be the length of the arc of the curve from the point $r(0) = (1, 0, 0)$ to $r(t)$. An arc length parameterization of the curve is given by

(A) $h(s) = (\cos(s^2), \sin(s^2), \sqrt{8}s^2)$ (B) $h(s) = (\cos(s), \sin(s), \sqrt{8}s)$
(C) $h(s) = (\sin(s/3), \cos(s/3), \sqrt{8}s/3)$ (D) $h(s) = (\cos(s/3), \sin(s/3), \sqrt{8}s/3)$
(E) $h(s) = (3\cos(s), 3\sin(s), 3\sqrt{8}s)$ (F) none of the above

Answer to 8: D

One has $r'(t) = (-\sin(t), \cos(t), \sqrt{8})$ so $|r'(t)| = \sqrt{1+8} = 3$ for all t. The arc length from $r(0)$ to $r(t)$ is

$$s(t) = \int_0^t |r'(w)|dw = \int_0^t 3w = 3t.$$

So $t = t(s) = s/3$, and we get the arc length paramaterization as $r(t(s)) = r(s/3)$.

(9) Let C be the curve parameterized by $r(t) = (1, t, \frac{t^2}{2})$. An equation in rectangular coordinates (x, y, z) for the osculating plane to C at the point $r(0) = (1, 0, 0)$ is:

(A) $z = 0$

(B) $y = 0$

(C) $x = 1$

(D) $x + y + z = 1$

(E) none of the above

Answer to 9: C

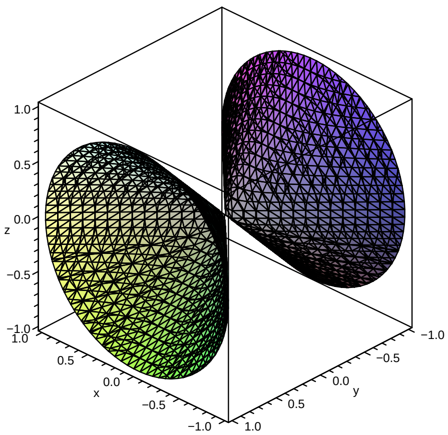
One definition of the osculating plane is that it is the plane through $r(0) = (1, 0, 0)$ which contains the lines with direction vectors $r'(0) = (0, 1, 0)$ and $r''(0) = (0, 0, 1)$. Since $(1, 0, 0)$ is a vector perpendicular to both these direction vectors, it is normal to the plane. An equation of the plane is thus

$$0 = ((x, y, z) - (1, 0, 0)) \cdot (1, 0, 0) = x - 1$$

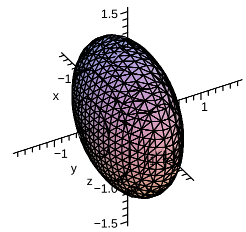
which leads to answer (C).

(10) Match up the graphs and equations

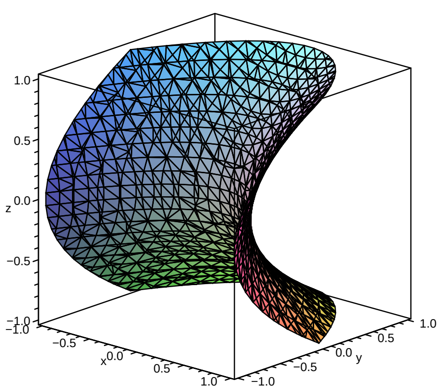
(A)



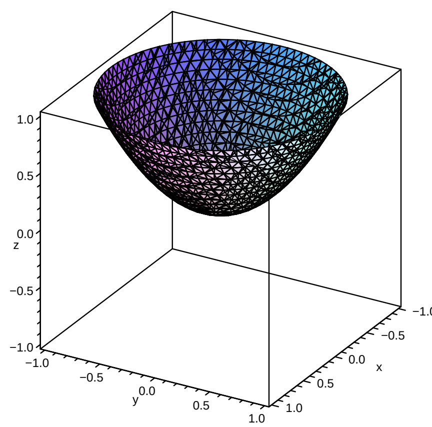
(B)



(C)



(D)



(i) $x^2 + 4y^2 + z^2 = 1$

(ii) $y = z^2 - x^2$

(iii) $y^2 = z^2 + x^2$

(iv) $z = (x^2 + y^2)$

Answer to 10: (i) B (ii) C (iii) A (iv) D