

## Math 114-001: Hour Exam 4

11 a.m., April 24, 2008

**Your Name:**

**Your T.A. and recitation time:**

**Instructions:** This exam is 45 minutes long. You can use one handwritten one-sided page of notes, but no books or calculators. **It is important that you show your work for each problem.** To receive credit for a problem, you must both indicate the correct answer and show plausible work justifying your answer. Include with your exam any blue books which have work you want us to consider, and be sure your name is on these blue books. There will be no partial credit. Do not come to the front of the class when the exam is over; we will pick up your exam from you.

**Multiple choice questions (10 points each).**

- (1) Suppose  $y(t)$  is a solution of differential equation  $\frac{dy}{dt} = te^y$  and that  $y(1) = 0$ . Find  $y(0)$ .

(A)  $\ln(\frac{1}{2})$

(B)  $\ln(2)$

(C)  $e^{3/2}$

(D)  $e^{1/2}$

(E)  $-\ln(\frac{3}{2})$

(F) none of the above

**Answer to 1: E**

Write the differential equation as  $\frac{dy}{e^y} = tdt$  and integrate:

$$\int e^{-y} dy = -e^{-y} + \text{constant} = \int t dt = t^2/2 + \text{constant}$$

So  $e^{-y(t)} = -t^2/2 + c$  for some constant  $c$ . When  $t = 1$  we have  $-1^2/2 + c = e^{-y(1)} = e^0 = 1$  so  $c = 3/2$ . Then when  $t = 0$  one has  $e^{-y(0)} = c = 3/2$  so  $y(0) = -\ln(3/2)$ .

(2) Suppose  $y(t)$  is a solution of the initial value problem

$$y''(t) + 2y'(t) + 2y(t) = 0 \quad \text{and} \quad y(0) = 2 \quad \text{and} \quad y'(0) = 1.$$

Find  $y(2\pi)$ .

(A)  $e$

(B)  $2$

(C)  $e^{-2\pi}$

(D)  $2e^{-2\pi}$

(E)  $\cos(2)$

(F) none of above

**Answer to 2:D**

The characteristic equation is

$$r^2 + 2r + 2$$

which has roots

$$(-2 \pm \sqrt{2^2 - 4 \cdot 2})/2 = -1 \pm \sqrt{-1}$$

Thus the fundamental solutions are

$$y_1(t) = e^{-t} \cos(t)$$

and

$$y_2(t) = e^{-t} \sin(t)$$

All solutions have the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some constants  $c_1$  and  $c_2$ . Observe that

$$y_1(2\pi) = e^{-2\pi} \cos(2\pi) = e^{-2\pi} \cos(2 \cdot 0) = e^{-2\pi} y_1(0)$$

and similarly

$$y_2(2\pi) = e^{-2\pi} y_2(0)$$

This implies that

$$y(2\pi) = e^{-2\pi} y(0)$$

So if  $y(0) = 2$  then  $y(2\pi) = e^{-2\pi} 2$ .

- (3) Let  $y(t)$  be the fraction of the population which has a cell phone. Suppose  $y(t)$  satisfies a logistic equation

$$y'(t) = y(t)(1 - y(t))/10.$$

Let  $t = 0$  stand for the start of the year 2000, and suppose  $y(0) = \frac{1}{2}$ . If  $t$  is measured in years, what fraction of the population will have a cell phone at the start of the year 2010?

- (A)  $\frac{3}{4}$ . (B)  $\frac{1}{1 + e^{-1}}$ .  
(C)  $\frac{1 - e}{e}$ . (D)  $1 - 2^{10}$ .  
(E) none of the above.

<b>Answer to 3: B</b>
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The solution of the logistic equation in this case is

$$y(t) = \frac{1}{1 + Ae^{-t/10}} \quad \text{when} \quad A = \frac{1 - y(0)}{y(0)}$$

as explained on pages 662 and 663 of the text. Since  $y(0) = 1/2$ , we find  $A = 1$ . Then

$$y(10) = \frac{1}{1 + e^{-10/10}} = \frac{1}{1 + e^{-1}}.$$

(4) Suppose  $v(t)$  is a solution of the initial value problem

$$v'(t) - 2tv(t) = 3t^2e^{t^2} \quad \text{and} \quad v(0) = 1.$$

Find  $v(-1)$ .

(A) 0

(B) 1

(C)  $2e$

(D)  $e$

(E)  $e^{-1}$

(F) none of the above.

<b>Answer to 4: A</b>
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We multiply by the integrating factor  $I = e^{\int -2tdt} = e^{-t^2}$  to have

$$\frac{d}{dt}(e^{-t^2}v(t)) = v'(t)e^{-t^2} + e^{-t^2}(-2t)v(t) = e^{-t^2}3t^2e^{t^2} = 3t^2.$$

Integrating both sides gives

$$e^{-t^2}v(t) = t^3 + c$$

for some constant  $c$ . Setting  $t = 0$  gives

$$e^{-0^2}v(0) = 0^3 + c$$

so

$$1 = v(0) = c.$$

Now letting  $t = -1$  gives

$$e^{-1}v(-1) = (-1)^3 + 1 = 0$$

so  $v(-1) = 0$ .

(5) Suppose that  $x(t)$  and  $y(t)$  are the populations of two species at time  $t$ , and that

$$\frac{dy}{dt} = 1000x - xy.$$

For which positive values of  $x$  and  $y$  is the species represented by  $x$  beneficial to the species  $y$  in the sense that the presence of  $x$  makes the population represented by  $y$  increase more rapidly than if  $x$  were equal to 0?

(A)  $y > 1000$  and  $x > 0$

(B)  $0 < y < 1000$  and  $x > 0$

(C)  $x > 1000$  and  $y > 0$

(D)  $0 < x < 1000$  and  $0 < y$

(E) none of above

<b>Answer to 5: B</b>
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Species  $x$  is beneficial to  $y$  exactly when  $1000x - xy > 1000 \cdot 0 - 0 \cdot y = 0$ . For positive  $x$  and  $y$  this occurs when  $1000 - y > 0$ , i.e. when  $0 < y < 1000$  and  $x > 0$ .

(6) Suppose that

$$y''(x) + y(x) = e^x + 1 \quad \text{and} \quad y(0) = 3/2 \quad \text{and} \quad y'(0) = 1/2.$$

The value of  $y(1)$  is:

(A)  $e/2$

(B) 1

(C)  $\frac{e}{2} + 1$

(D)  $\frac{e}{2} - 1$

(E) none of the above

<b>Answer to 6: C</b>
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The fundamental solutions of the associated homogeneous differential equation

$$y''(x) + y(x) = 0$$

are

$$y_1(x) = \cos(x)$$

and

$$y_2(x) = \sin(x)$$

We look for a particular solution of the non-homogeneous equation of the form

$$y_p(x) = A + Be^x$$

Then

$$y_p'(x) = Be^x = y_p''(x)$$

and

$$y_p''(x) + y_p(x) = A + 2Be^x = e^x + 1$$

This forces  $A = 1$  and  $B = 1/2$ . So the general solution is

$$y(x) = 1 + (e^x)/2 + c_1 \cos(x) + c_2 \sin(x).$$

Then

$$y'(x) = (e^x)/2 - c_1 \sin(x) + c_2 \cos(x).$$

So

$$y(0) = 3/2 = 1 + 1/2 + c_1$$

forces  $c_1 = 0$ , while

$$y'(0) = 1/2 = 1/2 + c_2$$

forces  $c_2 = 0$ . So  $y(x) = 1 + (e^x)/2$  and  $y(1) = 1 + e/2$ .

- (7) Suppose  $F_1(t)$  and  $F_2(t)$  are functions for which  $F_1'(t) = e^{-t}/t^2$  and  $F_2'(t) = e^t/t^2$ . Which of the following formulas give the general solution  $y(t)$  of the differential equation  $y''(t) - y(t) = 2/t^2$ ? In these formulas,  $c_1$  and  $c_2$  are arbitrary constants.

- (A)  $c_1e^t + c_2e^{-t}$  (B)  $(-F_1(t) + c_1)e^t + (F_2(t) + c_2)e^{-t}$   
 (C)  $(-F_1(t) + c_1)e^t + (-F_2(t) + c_2)e^{-t}$  (D)  $(F_1(t) + c_1)e^t + (F_2(t) + c_2)e^{-t}$   
 (E)  $(F_1(t) + c_1)e^t + (-F_2(t) + c_2)e^{-t}$  (F) none of the above

**Answer to 2: E**

The associated homogeneous differential equation is

$$y''(t) - y(t) = 0$$

This has characteristic equation  $r^2 - 1 = 0$ , which has roots  $r = \pm 1$ . So two fundamental solutions of the associated homogeneous differential equation are

$$y_1(t) = e^t$$

and

$$y_2(t) = e^{-t}.$$

Write these as  $y_1$  and  $y_2$ . The method of variation of parameters says that a particular solution of the non-homogeneous equation is given by

$$y_p = u_1y_1 + u_2y_2$$

for some functions  $u_1 = u_1(t)$  and  $u_2 = u_2(t)$  for which

$$0 = u_1'y_1 + u_2'y_2 = u_1'e^t + u_2'e^{-t}$$

and

$$2/t^2 = 1 \cdot (u_1'y_1' + u_2'y_2') = u_1'e^t - u_2'e^{-t}.$$

Adding and subtracting these two equations gives

$$2u_1'e^t = 2/t^2 \quad \text{and} \quad 2u_2'e^{-t} = -2/t^2.$$

Therefore

$$u_1' = e^{-t}/t^2 \quad \text{and} \quad u_2' = -e^t/t^2.$$

So

$$u_1(t) = \int e^{-t} \frac{dt}{t^2} = F_1(t) + c_1 \quad \text{and} \quad u_2(t) = - \int e^t \frac{dt}{t^2} = -F_2(t) + c_2$$

This leads to answer (E).

- (8) As  $k$  varies the curves  $C_k : y = kx^2$  form a family of parabolas in the  $x$ - $y$  plane. Find a family of curves  $D_c$  which is orthogonal to these parabolas, where  $c$  is a varying parameter.

(Hint: The slope of  $D_c$  at  $(x, y)$  should be the negative inverse of that of  $C_k$  at  $(x, y)$  if  $(x, y)$  is on both  $D_c$  and  $C_k$ . Try writing down this slope in a way which involves only  $x$  and  $y$  and not  $k$  or  $c$ .)

- (A)  $D_c : \frac{x^2}{2} + y^2 = c$       (B)  $D_c : 2cy = -\ln|x|$       (C)  $D_c : x + y = c$   
 (D)  $D_c : y = cx$       (E) none of the above

**Answer to 8: A**

The slope of the curve  $C_k$  at a point  $(x, y)$  on this curve is

$$\frac{dy}{dx} = 2kx = 2kx^2/x = 2y/x.$$

The curve  $D_c$  through  $(x, y)$  which is perpendicular to  $C_k$  should have slope

$$\frac{dy}{dx} = \frac{-1}{2y/x} = \frac{-x}{2y}.$$

Applying separation of variables to this equation gives

$$\int 2ydy = \int -x dx$$

This becomes

$$y^2 = -x^2/2 + c$$

for some constant  $c$ . So the curve  $D_c$  can be taken to be

$$\frac{x^2}{2} + y^2 = c.$$

(9) Suppose that the density of material inside the unit sphere in space is given by  $f(x, y, z) = z^2$ . What is the total mass inside the sphere?

(A)  $\frac{\pi}{9}$

(B)  $\frac{\pi}{4}$

(C)  $4\pi$

(D)  $\frac{4\pi}{9}$

(E)  $\frac{4\pi}{15}$

(F) none of the above

**Answer to 9: E**

Using spherical coordinates  $(\rho, \phi, \theta)$  we have  $z = \rho \cos(\phi)$ . So the mass is

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=1} \rho^2 \cos^2(\phi) \rho^2 \sin(\phi) d\rho d\phi d\theta$$

The integrand is the product of the three functions  $\cos^2(\phi)\sin(\phi)$ ,  $\rho^4$  and 1 which depend on  $\phi$ ,  $\rho$  and  $\theta$  alone. The limits for each variable do not depend on the other variables. So the integral breaks up into the product of

$$\int_{\theta=0}^{\theta=2\pi} d\theta = 2\pi$$

$$\int_{\phi=0}^{\phi=\pi} \cos^2(\phi)\sin(\phi)d\phi = \frac{-\cos^3(\phi)}{3} \Big|_{\phi=0}^{\phi=\pi} = -\frac{(-1)^3}{3} - \frac{-1}{3} = \frac{2}{3}$$

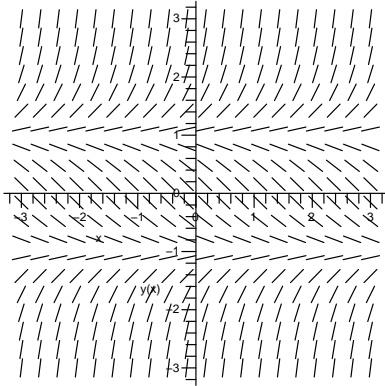
and

$$\int_{\rho=0}^{\rho=1} \rho^4 d\rho = \frac{\rho^5}{5} \Big|_{\rho=0}^{\rho=1} = \frac{1}{5}.$$

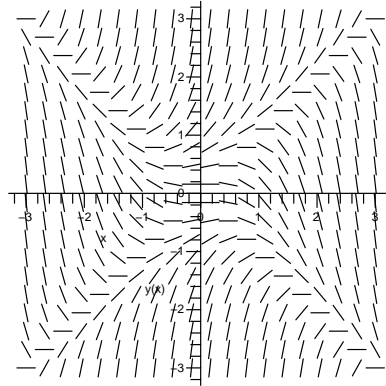
This leads to answer (E).

(10) Match up the differential equations with the direction fields.

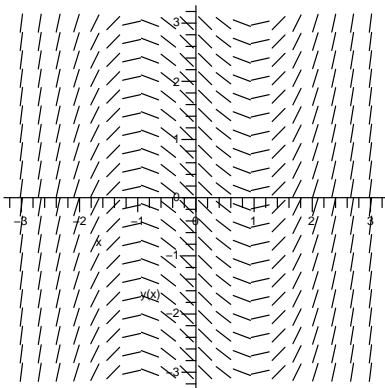
(A)



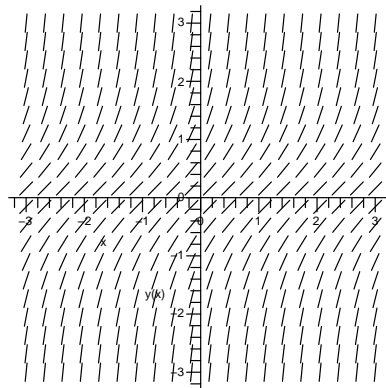
(B)



(C)



(D)



(i)  $y' = y^2 + 1$

(ii)  $y' = y^2 - 1$

(iii)  $y' = x^2 - 1$

(iv)  $y' = y^2 - x^2$

**Answer to 10: (i) D      (ii) A      (iii) C      (iv) B**

Equation (iii) is the only one in which the slope  $y'$  of the direction field at  $(x, y)$  depends only on  $x$ . In this case, the slopes should all be the same at points over a given point on the  $x$ -axis. Only graph (C) has this property, so (iii) corresponds to (C).

In equations (i) and (ii), the slope  $y'$  of the direction field at  $(x, y)$  depends only on  $y$ . So in these cases, the slope should not change as one moves horizontally in the  $x - y$  plane. Graphs (A) and (D) have this property, but (B) does not. So (B)

has to correspond to (iv). One can also see this from the fact that (iv) is the only case in which the slope  $y'$  is zero whenever  $y = \pm x$ .

We can distinguish (i) and (ii) by just noting that in (i),  $y'$  is positive when  $y = 0$  while in (ii),  $y'$  is negative when  $y = 0$ . So (i) corresponds to (D) and (ii) corresponds to (A).