

MATH 114 - EXTRA CREDIT PROBLEM # 1

ABSTRACT. You can turn in this problem anytime before the final exam. Work on extra credit problems will be used to decide cases in which a course grade is in between two letter grades.

1. TORQUE AS THE DERIVATIVE OF ANGULAR MOMENTUM

Suppose a particle of mass m is at position $r(t) = (x(t), y(t), z(t))$ at time t . In class we talked about the fact that the velocity of the particle at time t is

$$v(t) = r'(t) = (x'(t), y'(t), z'(t))$$

and the acceleration is

$$a(t) = v'(t) = (x''(t), y''(t), z''(t)).$$

The force vector acting on the particle is

$$F(t) = ma(t)$$

according to Newton's third law. In class we defined the angular momentum of the particle with respect to the origin to be the vector

$$A(t) = m(r(t) \times v(t)).$$

The torque acting on the particle is defined to be

$$\tau(t) = r(t) \times F(t).$$

- a. Let $i = (1, 0, 0)$, $j = (0, 1, 0)$ and $k = (0, 0, 1)$ be the vectors of length one in the direction of the coordinate axes. Write down the formula for $A(t)$ as a determinant and use this to write out $A(t) = (b_1(t), b_2(t), b_3(t))$ for some explicit functions $b_1(t), b_2(t), b_3(t)$. For instance, you should get

$$b_1(t) = m(y(t)z'(t) - z(t)y'(t)).$$

- b. By differentiating each component of $A(t)$, show that there is an equality of vectors

$$\begin{aligned} A'(t) &= (b'_1(t), b'_2(t), b'_3(t)) \\ &= m(r'(t) \times v(t)) + m(r(t) \times v'(t)) \\ &= m(v(t) \times v(t)) + m(r(t) \times a(t)). \end{aligned}$$

Here the final step uses $r'(t) = v(t)$ and $v'(t) = a(t)$.

- c. Show that for all vectors $v = (v_1, v_2, v_3)$ one has $v \times v = (0, 0, 0)$. Try doing this in two ways. The first involves using the geometric description of $v \times v$ and the fact that $\sin(0) = 0$. The second involves the explicit formula for how to compute $v \times v$.
- d. Apply part (c) in part (d) to conclude that

$$A'(t) = m(r(t) \times a(t)) = r(t) \times (ma(t)) = r(t) \times F(t) = \tau(t).$$

Thus the derivative of angular momentum is torque!