MATH 114 - EXTRA CREDIT PROBLEM # 4

ABSTRACT. You can turn in this problem anytime before the final exam. Work on extra credit problems will be used to decide cases in which a course grade is in between two letter grades.

Level curves and estimates for values of functions

Example 9 on page 930 of the Stewart text has to do with the level sets of a function $f : \mathbb{R}^2 \to \mathbb{R}$. The picture in figure 14 on this page of the text describes the level curves

$$C_r = \{(x, y) : f(x, y) = r\}$$

for $r = 50, 60, 70, 80$. Example 9 asks you to estimate the values of $f(1, 3)$ and $f(4, 5)$ given these level curves.

1. Explain why it is impossible to estimate $f(1, 3)$ from the given level curves if all we know is that f is a function from $\mathbb{R}^2$ to $\mathbb{R}$.
2. Suppose now that $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Define $h : \mathbb{R} \to \mathbb{R}$ to be the function defined by
$$h(x) = f(x, 3).$$
Use the ϵ-δ definition of continuity to show that h must be a continuous function of x.
3. Suppose f and h are as in problem # 2. Using figure 14 on page 930 of the Stewart text, explain why there are numbers x_0 and x_1 with the following properties:
   a. $0 < x_0 < 1 < x_1 < 2$.
   b. $f(x_0, 3) = h(x_0) = 70$ and $f(x_1, 3) = h(x_1) = 80$.
   d. There is no number x such that $x_0 < x < x_1$ and $f(x, 3) = h(x) = 70$.
   c. There is no number x such that $x_0 < x < x_1$ and $f(x, 3) = h(x) = 80$.
4. The intermediate value Theorem says that if $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, and if $a < b$ are real numbers, then for every number t between $h(a)$ and $h(b)$ which is not equal to either $h(a)$ or $h(b)$, there is a number c such that $a < c < b$ and $h(c) = t$. With the assumptions and notations of problems # 2 and # 3, use this to show that for all x with $x_0 < x < x_1$ we have to have
$$70 < h(x) = f(x, 3) < 80.$$
In particular, $70 < f(1, 3) < 80$ since 1 lies between x_0 and x_1. This is the most one can say from the information in Example 9 on page 930 of the text.

   Hint: We know from problem # 3 that there is no x such that $x_0 < x < x_1$ and $h(x) \in \{70, 80\}$. So if the statement we want to show is false, there has to be an x for which $x_0 < x < x_1$ and either $h(x) < 70$ or $h(x) > 80$. Suppose $h(x) < 70$, the other case being similar. Apply the intermediate value theorem with $a = x$, $b = x_1$ and $t = 70$.