

## MATH 114 - EXTRA CREDIT PROBLEM # 6

ABSTRACT. You can turn in this problem anytime before the final exam. Work on extra credit problems will be used to decide cases in which a course grade is in between two letter grades.

### Calculus and criminology

This problem is about applying Newton's Law of cooling to determine the time a murder took place. The perpetrator of the crime took math 114, and remembered that the law of cooling could be used to determine how long a body had been cooling given the temperature of the environment. However, they didn't do this extra credit problem, so they decided to try turning up the thermostat in the room where the murder took place right at the time of the murder in order to disguise the time this occurred. This problem is about what a detective who did do the extra credit for math 114 could do to find the time of the crime anyway.

Suppose the detective knows the following information:

1. The temperature of the room just before the murder occurred was  $b_0$  degrees Celsius.
2. When the murder occurred, the perpetrator turned the thermostat up to  $b_1$  degrees, where  $b_1 \geq b_0$ .
3. Fix the time of the murder at  $t = 0$  and the time the detective arrives at the crime scene as  $t = t_0$ . The detective wants to know what  $t_0$  is. Suppose the temperature  $b(t)$  of the room at time  $t$  satisfies Newton's law of cooling

$$(0.1) \quad b'(t) = c_0(b(t) - b_1) \quad \text{and} \quad b(0) = b_0$$

where  $c_0 < 0$  is a constant. Explain why it is the constant  $b_1$  which enters into the right side of the first equality in (0.1). How would the detective estimate  $c_0$  from observing how much the room cools down for one minute after they arrive at the crime scene at time  $t = t_0$ ? (Hint:  $b(t+1) - b(t)$  is nearly  $b'(t) \cdot 1 = c_0(b(t) - b_1)$ , and  $b_1$  is the temperature to which the thermostat has been set.)

4. The temperature  $f(t)$  of the body at time  $t$  satisfies Newton's law of cooling in the form

$$(0.2) \quad f'(t) = c_1(f(t) - b(t)) \quad \text{and} \quad f(0) = 37$$

where 37 degrees Celsius is 98.6 degrees Fahrenheit and  $c_1 < 0$  is a constant. Explain why  $b(t)$  enters this way into the right side of (0.2). How would the detective estimate  $c_1$  by observations made for one minute after arriving at the crime scene at time  $t = t_0$ ?

Given this initial information, analyze  $f(t)$  using the following steps.

- A. Find a formula for  $b(t)$  using (0.1).
- B. Plug this formula into (0.2) and solve for  $f(t)$  using the theory of integrating factors discussed in section 10.6 of the text.
- C. Show that your solution for  $f(t)$  has one of the following two forms:

$$(0.3) \quad f(t) = b_1 + a_0 e^{c_0 t} + a_1 e^{c_1 t} \quad \text{if} \quad c_0 \neq c_1$$

$$(0.4) \quad f(t) = b_1 + (a_0 t + a_1) e^{c_0 t} \quad \text{if} \quad c_0 = c_1$$

where  $a_0$  and  $a_1$  are constant. Show that if  $a_0$  and  $a_1$  are both zero, then  $f(t) = b_1$  is a constant and  $f'(t) = 0$ . Deduce that this is so if and only if  $f(0) = b_0 = b_1$ ; you may find it

useful to let  $t \rightarrow +\infty$  to prove this. Explain physically what  $f(0) = b_0 = b_1$  would mean in terms of the initial and equilibrium temperatures of the room and the initial temperature of the body. Why would one not expect to be able to deduce the time of the crime if  $f(0) = b_0 = b_1$ ?

- D.** Now suppose it is **not** the case that  $f(0) = b_0 = b_1$ , so at least one of  $a_0$  and  $a_1$  are not zero. Use part (C) to show

$$(0.5) \quad f'(t) = a_0 c_0 e^{c_0 t} + a_1 c_1 e^{c_1 t} \quad \text{if } c_0 \neq c_1$$

$$(0.6) \quad f'(t) = a_0 t c_0 e^{c_0 t} + a_0 e^{c_0 t} + a_1 c_0 e^{c_0 t} = (a_0 c_0 t + a_0 + a_1 c_0) e^{c_0 t} \quad \text{if } c_0 = c_1$$

Show that there is a choice of  $i = 1$  or  $i = 2$  such that  $e^{-c_i t} f'(t)$  has the form  $n_0 + n_1 e^{nt}$  if  $c_0 \neq c_1$  or  $n_0 + n_1 t$  if  $c_0 = c_1$  where at least one of  $n_0$  or  $n_1$  is not 0.

- E.** Suppose as in part (D) that it is not the case that  $f(0) = b_0 = b_1$ . Conclude from part (D) that  $f'(t) = 0$  for at most one value  $t_1$  of  $t$ , and that if such a  $t_1$  exists then  $f'(t)$  has opposite signs for  $t > t_1$  and for  $t < t_1$ . (In fact, if  $n_1 = 0$  in part (D) then  $n_0 \neq 0$  and no such  $t_1$  exists.) Explain why this implies that if  $b_0 \neq b_1$ , then  $f(t)$  takes on the same value  $z$  for at most two distinct values  $t_{z,1}$  and  $t_{z,2}$  of  $t$ , and that if two such distinct values exist, then  $f'(t)$  has different signs at  $t_{z,1}$  and  $t_{z,2}$ . Explain why this means that if the detective observes  $f(t_0)$  on arriving at the crime scene, then unless  $f(0) = b_0 = b_1$ , the detective can deduce that  $t_0$  must be one of only two possible values  $t_{z,1}$  and  $t_{z,2}$ . Furthermore, show that if they can distinguish whether it is time  $t_{z,1}$  or time  $t_{z,2}$  by whether they body is still cooling or whether it is heating up. Discuss what sort of numerical technique you might use to find the possibilities for  $t_0$  given  $f(t_0)$ .