MATH 210, PROBLEM SET 1

DUE IN LECTURE ON WEDNESDAY, SEPT. 15.

1. Games with saddlepoints

Before trying to do these problems, read section 15.1 of the "For all practical purposes" text, which I will refer to as FAPP. On the next page I summarize in more mathematical terms the meaning of saddlepoints of two person zero sum games. You may also find it useful to look at the answers to exercises #1, #3 and #5 on page 584 of FAPP which are given in appendix A of FAPP (see page A-35).

Please briefly explain your reasoning for each problem; the credit I can give depends on your reasoning.

(1) Do #1 of the "Skills check" problems in Chapter 15 of FAPP (page 581).

(2) Do #2 of the "Skills check" problems in Chapter 15 of FAPP (page 581).

(3) Do #4 of the "Skills check" problems in Chapter 15 of FAPP (page 582).

(4) Do #5 of the "Skills check" problems in Chapter 15 of FAPP (page 582).

2. Games with mixed strategies.

To understand this topic, I suggest reading pages 21 to 33 of the book "How math can save your life" which is available on gigapedia, as well as section 15.2 of FAPP. See the course guide on the math 210 web page for how to use gigapedia. I think pages 31 to 33 of "How math can save your life" are actually clearer than FAPP when it comes to describing how to choose mixed strategies.

(1) Do #16 of the "Skills check" problems in Chapter 15 of FAPP (page 583).

(2) Describe a situation in the news, at Penn or in your own experience which can be viewed as a two-person two-option zero sum game. Formulate and explain the payoff matrix for one player in this game and determine what the optimal strategies of the players should be. Comment on the significance of your answer. (I will ask people who include amusing or otherwise unusual answers to this problem whether they would like their answers posted on a link of the course page. It's fine if you'd rather not have such answers posted, though.)

The theory of saddlepoints

Suppose that two players, named I and II, play a zero-sum game in which player I has n options while player II has m options. If player I plays option i and player II plays option j, the payoff to player I is some number $a_{i,j}$, while the payoff to player II is $-a_{i,j}$. (The game is zero sum because the payoff to each player is exactly the negative of the payoff to the other.)

The Payoff matrix for player I is then

$$A = (a_{i,j})_{1\le i\le n, 1\le j\le m}.$$

Rows in this matrix are options for player I, while columns are options for player II.

The maximin of this matrix is

$$\text{maximin}(A) = \max_{1\le i\le n}(\min_{1\le j\le m}\ a_{i,j}) \tag{2.1}$$

This can be computed by first finding

$$A_{row}(i) = \min_{1\le j\le m}\ a_{i,j} = a_{i,j(i)} = \text{a minimal entry in row } i$$

for some integer $j(i)$ depending on i. There may be more than one $j(i)$ which works. One then finds

$$\text{maximin}(A) = \max_{1\le i\le n}\ A_{row}(i) = a_{i_1,j(i_1)}$$

for some i_1.

Player II would like to minimize the payoff to player I of playing the game, since this maximizes the payoff to them. The significance of $A_{row}(i)$ is that if player II knows that player I is going to play option i, they should choose a stategy $j = j(i)$ for themselves which minimizes $a_{i,j}$ as j runs from 1 to m. The payoff to player I of playing option i is then $A_{row}(i)$.

If player I realizes that someone is going to leak their choice of strategy to player II, then they should choose i so that $A_{row}(i)$ is as large as possible. This leads to the payoff maximin(A) to player I (and payoff $-$maximin(A) to player II). One should think of this number as the best player I can do if once they choose an option, this choice is told to player II, so that player II can best counter the choice made by player I.

The minimax of the matrix A is

$$\text{minimax}(A) = \min_{1\le j\le m}(\max_{1\le i\le n}\ a_{i,j}) \tag{2.2}$$

This can be computed by first finding

$$A_{col}(j) = \max_{1\le i\le n}\ a_{i,j} = a_{i(j),j} = \text{a maximal entry in column } j$$

for some integer $i(j)$. One then finds

$$\text{minimax}(A) = \min_{1\le j\le m} A_{col}(j) = a_{i(j_2),j_2}$$

for some j_2. This number represents the minimal payoff to player I (and thus the maximal payoff to player II) which player II can force if the option that player II picks is communicated to player I so that player I can best counter it.

One says A has a saddlepoint if

$$\text{maximin}(A) = \text{minimax}(A)$$

In this case, suppose maximin(A) $= a_{i_1,j(i_1)}$ and minimax(A) $= a_{i(j_2),j_2}$ as above. Then

$$\text{maximin}(A) = a_{i_1,j(i_1)} \le a_{i_1,j_2} \le a_{i(j_2),j_2} = \text{maximin}(A)$$

by the definitions of $j(i_1)$ and $i(j_2)$. But we are supposing A has a saddlepoint, so the far left and far right terms are equal, and we get

$$\text{maximin}(A) = a_{i_1,j(i_1)} = a_{i_1,j_2} = a_{i(j_2),j_2} = \text{maximin}(A).$$

We could have chosen $j(i_1)$ to be any j which minimizes $a_{i,j}$ as j ranges over $1 \leq j \leq m$, so this shows we might as well have chosen $j(i_1) = j_2$. Similarly we could choose $i(j_2) = i_1$.

The upshot is that if A has a saddlepoint, then players I and II should pick options i_1 and j_2 respectively. Player I is then guaranteed a payoff of $\text{maximin}(A) = \text{minimax}(A)$ and player II can ensure that the payoff is not larger than this. Neither player has an incentive to make another choice, since if they did, the other player could benefit at their expense. For instance, if player I picked some other option i_1', then

$$A_{row}(i_1) \geq A_{row}(i_1') = a_{i_1',j(i_1')}$$

and if this is a strict inequality, then player II could make things worse for player II by picking option $j(i_1')$ rather than j_2.

If A does not have a saddlepoint, there need not be one best option for each player. They may raise their expected returns by choosing randomly between their options giving each one a certain probability of being chosen. This is the subject of §15.2 of FAPP.