MATH 210, PROBLEM SET 2

DUE IN LECTURE ON WEDNESDAY, SEPT. 22.

A three option two person zero sum game.

The rock-paper-scissors game has two players who simultaneously choose between three options; rock, paper or scissors. If they choose the same option, the outcome is a draw, resulting in a payoff of 0 to both players. Otherwise, the payoff to each player is either 1 or -1 according to the following rule: rock beats scissors, scissors beats paper and paper beats rock.

Problems:

1. Write down the payoff matrix for the first player (player I) of this game. There should be three rows and three columns corresponding to the choices of the two players.

2. Find the maximin and minimax of this payoff matrix from the point of view of the first player. Is there a saddle point?

3. Suppose that the two players pursue mixed strategies represented by the vector $P = (p_1, p_2, p_3)$ for player I and the vector $Q = (q_1, q_2, q_3)$ for player II, where

$$0 \leq p_i \leq 1 \quad \text{and} \quad 0 \leq q_j \leq 1$$

for all i and j and

$$p_1 + p_2 + p_3 = 1 = q_1 + q_2 + q_3.$$

3a. Suppose that player I picks $(p_1, p_2, p_3) = (1/3, 1/3, 1/3)$. Show that for all choices of $Q = (q_1, q_2, q_3)$ by player II, the expected payoff of the game to player I is 0. Recall that the expected payoff is the sum over all possible pairs of choices of the probability of that choice times the payoff associated to this pair. You may find it useful when computing this expected payoff to group together the terms which correspond to a fixed choice by player II.

3b. Suppose now that player I picks a vector $P = (p_1, p_2, p_3)$ as above which is different from $(1/3, 1/3, 1/3)$. How could player II pick $Q = (q_1, q_2, q_3)$ to take advantage of $P \neq (1/3, 1/3, 1/3)$ in order to make the payoff to player I less than 0? Can player II do this by choosing a pure strategy, i.e. one for which $q_j = 1$ for some j?

3c. Let $E(P, Q)$ be the expected value of the payoff to player I from if player I chooses P and player II chooses Q as above. Conclude from problems (3a) and (3b) that

$$v_I = \max_P(\min_Q \ E(P, Q)) = 0$$

where P and Q ranges over all allowable choices.

3d. Using similar ideas, show that

$$v_{II} = \min_Q(\max_P\ E(P,Q)) = 0$$

and that this value is achieved when player II picks $Q = (1/3, 1/3, 1/3)$. Conclude that the value of the game is 0 and that the best mixed strategies for each player are to choose each option a third of the time.

4. Show that the rock-paper-scissors game arises in from the following scenario.

A movie critic (player I) writes a review of a film and has must give it either 1, 2 or 3 stars. A movie viewer has seen the film before reading the critic's review, and has also arrived at their own rating of the film as being worth 1, 2 or 3 stars. The viewer reads the critic's review, and does the following:

i. If the critic and the viewer agree, the viewer does not change their opinion.
ii. If the number of stars given to the movie by the critic and the viewer differ by 1, then the viewer decides that the higher of the two ratings (= more stars) is correct.
iii. If the ratings of the critic and the viewer differ by two stars, then the viewer loses confidence in the movie, and the viewer takes the lower of the two ratings.

The payoff to the critic is 0 if the critic and the viewer gave the same number of stars initially. Otherwise, the payoff to the critic is +1 if the viewer changes his rating to that of the critic, and it is −1 if the viewer does not change to the critic's rating.

4a. Show that the payoff matrix which results is the same as the payoff matrix for the rock-paper-scissors game when the three options rock/paper/scissors are matched up appropriately with 1, 2, and 3 star reviews.

4b. In a zero sum game, the payoffs to the viewer should be the negative of the payoffs to the critic. How would you explain this given the above description of they payoffs? In other words, why might the zero sum condition apply?

4c. If we treat this as a zero sum game, what are the optimal mixed strategies for the critic and the viewer? From the point of view of the critic, under what circumstances would it makes sense to choose this optimal strategy. In other words, describe a scenario under which the critic would want to choose a set $P = (p_1, p_2, p_3)$ of frequencies for 1, 2 and 3 star reviews which is best from the point of view of the minimax theorem. What would the critic be protecting themselves from by doing that? (Hint: Suppose that many readers reading the critic's reviews. Show that if a third of the readers start off giving a movie one star, a third give it two stars and a third give it three stars, then the payoff to the critic does not depend on how the critic chooses his strategy. Conversely, when does it matter most that the critic choose his optimal game theory strategy?)

5. Can you think of another situation in real life involving two players who end up playing a zero sum rock-paper-scissors game? I will ask people who submit unusual suggestions whether they'd be willing to let me post their examples on the course web page.