MATH 210, PROBLEM SET 3

DUE IN LECTURE ON WEDNESDAY, SEPT. 29.

A multi-option two-person game.

The object of this set of exercises is to find the optimal strategy in a two-person zero sum game in which the first player has two options and the second has three options. We will find the best strategy for player I in two ways. The first method maximizes the minimum of three linear forms. The second method uses the linear programming method described in class.

The two players in this game are political opponents. It has just come out that Player II once described her experiences with witchcraft on a talk show. The two players have to decide how to treat this in the hundreds of ads they will run before the election. Player I can run two kinds of ads:

1. An ad consisting of a clip of Player II discussing her interest in witchcraft, or
2. An ad which talks about the economy rather than witchcraft.

Player II can run three kinds of ads:

1. An ad which ignores the witchcraft issue entirely.
2. An ad which describes her witch experiences as a "youthful indescretion."
3. An ad which counterattacks by questioning whether Player I is discriminating against witches.

Player I wants to choose the relative proportions of the two kinds of ads they run so as to maximize the expected benefit of the ads against any strategy by Player II. The payoff matrix for Player I is as follows:

	Player II option 1 "ignore"	Player II option 2 "youthful indiscretion"	Player II option 3 "witches are people too"
Player I option 1 run witch clip	8	6	-6
Player I option 2 run economy clip	0	1	2

The payoffs to player II are the negatives of the payoffs to player I.

Problems:

1. Based on this payoff matrix, does the "witch's rights" counterattack work against a pure strategy by player I of always running the witch clip? How would you explain the payoff when player I does not run the witch clip but player II bringing up witches rights?

2. Does one strategy for either player dominate another strategy by the same player? (Note that if this happens, one can reduce the complexity of the game by eliminating dominated strategies.)

3. Suppose that player I chooses their option #1 with frequency p and their option #2 with frequency $1-p$. We discussed in class why the minimal payoff to player I of this strategy against any strategy by player # 2 occurs against some pure strategy by player II. Explain why this means that player I should choose p to be a number in the interval $0 \le p \le 1$ where the function

$$h(p) = \min\{8p, 6p + 1 - p, -6p + 2(1-p)\} \tag{0.1}$$

attains its maximum. Draw the graphs of the three linear functions appearing on the right side of (0.1). Find what value of p player I should pick and explain your reasoning.

4. We are now going to find the solution a second way using linear programming. The payoff matrix is:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} = \begin{pmatrix} 8 & 6 & -6 \\ 0 & 1 & 2 \end{pmatrix}.$$

Recall that the first step in this method is to add a positive constant to every entry of A in order to obtain a matrix with all positive entries. Choosing the constant to be $c = 7$, one gets the matrix

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{1,1} & \tilde{a}_{1,2} & \tilde{a}_{1,3} \\ \tilde{a}_{2,1} & \tilde{a}_{2,2} & \tilde{a}_{2,3} \end{pmatrix} = \begin{pmatrix} 15 & 13 & 1 \\ 7 & 8 & 9 \end{pmatrix}.$$

The associated linear programming problem is to find a vector $s = (s_1, s_2) \ge (0,0)$ such that $s\tilde{A} \ge (1,1,1)$ and $f(s) = s_1 + s_2$ is as small as possible. Write down

$$s \ge (0,0) \quad \text{and} \quad s\tilde{A} = (s_1, s_2) \cdot \begin{pmatrix} 15 & 13 & 1 \\ 7 & 8 & 9 \end{pmatrix} \ge (1,1,1)$$

as a system of five linear inequalities in the unknowns s_1 and s_2.

5. Continuing with the notation in problem #4, there are now five inequalities to be satisfied by s_1 and s_2. Draw the lines in the s_1-s_2 plane which form the borders of the allowed region defined by these inequalities. In class we talked about the fact that there must be a solution to the linear programming problem in #4 which is a vertex of the region described by the above inequalities. Each such vertex must be the intersection point of two of the 5 lines above. How many of these intersections describe vertices which are in the allowed region? (Hint: On your graph of the lines associated to the inequalities, write down the coordinates of the intersection points of the lines. Then test these points to see if they satisfy all the inequalities in the set of 5 constaints. This will check whether your picture of the lines is correct - an alternative is to check it with Maple!)

6. For each of the vertices you find in problem #5 which are in the allowed region, calculate the objective function

$$f(s) = s_1 + s_2.$$

Determine which of these $s = (s_1, s_2)$ minimizes $f(s)$, and check that

$$x^* = (p, 1-p) = \frac{1}{f(s)} s = \left(\frac{s_1}{f(s)}, \frac{s_2}{f(s)} \right)$$

gives the same optimal strategy for player I found in problem #3.

7. So far we have been discussing two-person zero sum games in which player I has some number n of options and player II has some number m of options. Can you think of a scenario in which player I has $n = 2$ options, but the options for player II are described by the choice of a real number r in the interval $0 \le r \le 1$? The payoff matrix for player I for such a game would have this form:

	Player II option r $0 \le r \le 1$
Player I option 1	$a_1(r)$
Player I option 2	$a_2(r)$

in which $a_1(r)$ and $a_2(r)$ are some functions of r. Assume that $a_1(r)$ and $a_2(r)$ are integrable functions of r. A mixed strategy for player I would play option #1 with probability p and option #2 with probability $1 - p$. A mixed strategy for player 2 would play option r with probability $q(r) \ge 0$, where we assume $q(r)$ is a continuous function of r such that

$$\int_{r=0}^{r=1} q(r)dr = 1$$

since this integral represents the probability of playing some strategy. Explain why the integral

$$E(p, q) = \left(p \int_{r=0}^{r=1} a_1(r)q(r)dr\right) + \left((1 - p) \int_{r=0}^{r=1} a_2(r)q(r)dr\right)$$

is the expected payoff for player I of this choice of p and the function $r \to q(r)$.

Extra Credit

These problems can be turned in at any time during the semester

EC1. Show that in any two-person zero sum game in which Player I has two options and Player II has three options, the optimal strategy $y^* = (y_1, y_2, y_3)$ for player II has the property that one of y_1, y_2 or y_3 is 0. (Hint: Consider the linear programming method for finding $y^* = t/g(t)$, where $t = (t_1, t_2, t_3)$ and $g(t) = t_1 + t_2 + t_3$. The vertices of the allowed region must be the **unique** solution to some system of equations defined by the planes which bound the region. How many of these planes are there, and how many are coordinate planes defined by setting some t_j equal to 0?)

EC2. Find the optimal strategy y^* for the game described in problems #1 - #6 the first part of this problem set.

EC3. With the notations of problem #7, suppose player I picks a value for p. Explain why the infimum of $E(p, q)$ over all choices of the continuous function $r \to q(r)$ will be equal to

$$f(p) = \text{infimum}_{0 \le r \le 1}\{pa_1(r) + (1-p)a_2(r)\}.$$

For a given r, the graph of $z = pa_1(r) + (1-p)a_2(r)$ as p ranges over $0 \le p \le 1$ is a line in the p-z plane. Explain why $f(p)$ is the infimum of the heights over p of all these lines as r varies over $0 \le r \le 1$. Then explain why player I would like to find a p which maximizes $f(p)$.