MATH 210, PROBLEM SET 4

DUE IN LECTURE ON FRIDAY, OCT. 8, WHICH IS THE DATE OF THE MID-TERM

1. Games with complete information

1. A game with complete information is one in which two players take turns choosing positions and have complete information about the sequence of positions which have come before their move. The players know at all times what options they have for their next move and the rule for deciding whether a sequence of moves leads to a loss, win or draw for them. For this reason, random outside events cannot be allowed, since these would change either the allowed positions or the consequences of a move by a player. Also, the players should have no uncertainty about whether they can in fact succeed in making a certain move, even when that move would lead to a loss for them.

   Which of the following games are games with complete information? Explain your reasoning in each case. You may find it useful to compare the cames to tic-tac-toe and chess, which are games with complete information, and to rummy, which is not.
   a. Checkers.
   b. Tennis.
   c. Monopoly.
   d. Mancala (see http://www.greenmountainblocks.com/gamerules.html for a description of the rules; use only the "general rules.")
   e. Going out on a date.

2. A strategy for a player in a game which has complete information is a rule which assigns to list of the previous moves in the game up to some point which more the player should make next. Consider the game of tic-tac-toe. Let's say that the allowed positions which a player can choose starting from a given position consist of adding either an X (if they are the first player) or an O (if they are the second player) to an open square on the tic-tac-toe board.
   a. Show that if $1 \leq j \leq 9$ then there are exactly $9 \cdot 8 \cdots (9 - j + 1)$ distinct ordered sequences of j allowed moves by the two players. Explain your reasoning. Ideally, try to write up your answer formally using induction on j. To do this, first show that the answer is correct for $j = 1$. Then show that if the answer is correct for some choice of $j = j_0$ with $1 \leq j_0 < 9$, then it is also true for $j = j_0 + 1$.
   b. Suppose $0 \leq j \leq 8$. Show that after j moves have been made, the next player has exactly $9 - j$ possible moves.

2. Partial conflict games

Suppose that the majority and minority part in congress can each decide whether to insist on it's own priorities or compromise with the other side. Each party ranks its preferences for each outcome by the numbers $1, 2, 3, 4$, with 4 being the best outcome for them and 1 being the worst. The majority party chooses its payoff ranking by treating the options likes a game of chicken, in which compromising is like swerving and insisting on one's own priorities amounts to not swerving. The minority party chooses its payoff ranking by treating the options as in the prisoner's dilemma, with compromising being the choice to cooperate and insisting being the choice to compete.

3. Write down the matrix which described the payoff rankings of each player for the four possible outcomes of their choices. Each entry of the matrix should be an ordered pair of numbers chosen from $\{1, 2, 3, 4\}$.
4. Find all of the Nash equilibria for the matrix of payoffs in problem # 3. At a Nash equilibria, neither player has a reason to unilaterally change their strategy. These equilibria therefore represent choices of strategies which are likely to persist once started. What does your answer predict about how congress will behave? Does this describe current congressional behavior?

3. Voting strategies

5. Describe a situation in your own experience or otherwise of interest to you in which three or more parties have had to vote on three or more options $x_1, \ldots, x_m$. Suppose that the ordering of the options given by $x_1 \geq x_2 \geq \ldots \geq x_m$ represents the status quo. The option which recieves the largest number of votes wins, with ties being decided by picking which of the options having the same maximal number of votes is highest on the list represented by the status quo. (When $m = 3$, this just the system described in the book when there are three voters.) Write down the preference matrix which applies, and analyze what strategic voting strategies the voters could use. Do your predictions reflect the reality of the situation you describe?

4. Truels

6. Do exercise # 32 for section 15.4 of the text. This problem appears on pages 504 and 505 of the 8^{th} edition of the FAPP text.

Extra Credit

This problem can be turned in at any time during the semester

1. A game with complete information is played by two players taking turns choosing elements of a set P of possible positions in the game. A strategy for each player consists of specifying a rule for what to do whenever it is their turn based on the moves which have taken place up to that time. Suppose P is a non-empy finite set.
   a. Suppose $j \geq 1$, and that exactly j moves have already taken place. Show that the number of ways the first j moves could have unfolded is $(\#P)^j$. Show that this count is also correct if $j = 0$.

   Comments: When $j \geq 1$, many sequences $(p_1, \ldots, p_j)$ of moves might be ruled illegal by the rules of the game, but we now allow them to be made. Illegal moves just lead to a loss by the first player to have made one. We are allowing illegal moves of this kind in order to make the counting simpler. In problem #2 above about tic-tac-toe, we prevented the players from making illegal moves, and that leads to a move complicated count.

b. Suppose that $j \geq 0$ moves have taken place. A strategy for the next player to move consists of giving a rule for what to do next for every possible way that the game could have proceeded up to this point. Show that for $j \geq 1$, such a specification amounts to giving a function $f : P^j \to P$, where P^j is the set of all j-tuples $(p_1, \ldots, p_j)$ of elements of P. For $j = 0$, explain why this amount to simply given an element of P, namely the first move in the game. Show that for all $j \geq 0$, the number of ways of specifying what to do is $(\#P)^{(\#P)^j}$.

c. Suppose that the rules of the game require it to end after exactly N moves. A memoryless strategy for the first player consists of giving a rule for each even integer $j \geq 0$ about how to move based on every possible sequence of the first j moves, without taking into account the fact that strategic choices for small j will restrict the possible sequences of moves which can come up. Show that there are

$$\prod_{0 \leq j \leq N-1,\ j\ even} (\#P)^{(\#P)^j} = (\#P)^M$$

such memoryless strategies, where

$$M = \sum_{0 \leq j \leq N-1,\ j\ even} (\#P)^j = 1 + (\#P)^2 + (\#P)^4 + \cdots + (\#P)^{2\ell} = \frac{(\#P)^{2\ell+2} - 1}{(\#P)^2 - 1}$$

when 2ℓ is the largest even integer $\leq N - 1$.

d. **(Harder!)** Suppose that we take into account the fact that once player I has specified a strategy for the moves numbered $1, 3, \ldots, 2k-1$, this restricts the possible ways the first $2k$ moves in the game could unfold. Count the number of ways the first $2k$ moves could have unfold given the strategy of the first player for each of moves $1, 3, \ldots, 2k-1$. Then use this to count to determine how many possible strategies player I has for their move after the player II has made the $2k^{th}$ move of the game if player I remembers their strategy in earlier moves. Call these "strategies with memory". Finally, count how many strategies with memory there are for player I if the game has total of exactly $N \geq 1$ moves.

Comments: The counting becomes yet more complicated if one prevents the players from making illegal moves - you might try to do this for tic-tac-toe. Memoryless strategies come up in the real world in the following way. Think of a strategy for player I as consisting of briefing books for what to do on move #1, on move #3, and so on. When it comes time to decide what to do on move $\#(2k+1)$, player I may either still have access to the briefing books for moves #1 through $\#(2k-1)$, or these briefing books may have been burned in the interests of security. If the books aren't burned, then player I will need a less extensive briefing book about what to do on move $\#(2k+1)$, since many sequences of earlier moves could not have unfolded. But if the earlier books are immediately burned after they are used, player I will need a move extensive briefing book for what to do on move $\#(2k+1)$. One can also think of this situation as arising in diplomacy. Country I might being in a new person to make the next move at each point in some sequence of negotiations with Country II. If the new person brought in has no knowledge of the strategies of earlier representatives of country I, then they will need a memoryless strategy for country I.