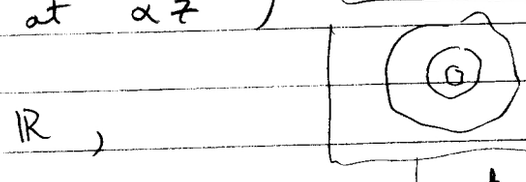
Two note number theory

I. Droste music
II. Enumerating Ragas

Thank: Martin Vuth
Modul
Fred Baker
Martijn Feleus

Droste picture: A Picture in $\mathbb{C}$ so that for some $\alpha \in \mathbb{C}^*$,

$$\begin{pmatrix}\text{color of pixel}\\ \text{at } z\end{pmatrix} = \begin{pmatrix}\text{color of pixel}\\ \text{at } \alpha z\end{pmatrix} \quad \text{for all } z \in \mathbb{C}$$



Droste music. For some $1 < \alpha \in \mathbb{R}$,

∴ a note starts at time t $\Rightarrow$ another note must start at αt

not iff: Don't want ~~•••|—|—~~
0 αt_1 t_1

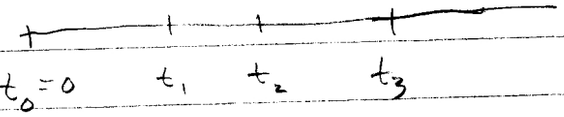
Harmonies: T = all t such that a note is played at time t

$= \{t_0, t_1, t_2, \ldots\}$

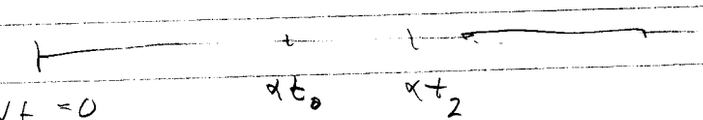
blows out speakers!

$$T \supseteq \alpha T = \{\alpha t_0, \alpha t_1, \alpha t_2 \ldots$$

first player: example

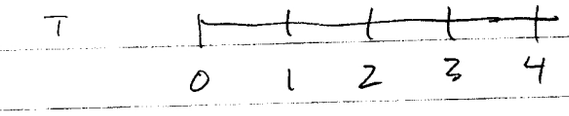


second player

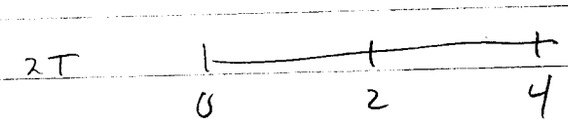


When second player plays same score at $\frac{1}{\alpha}$ the speed of the first, each time he/she plays a note the first player does also.

Ex: $T = \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$ | Proste for all $0 < \alpha \in \mathbb{Z}$



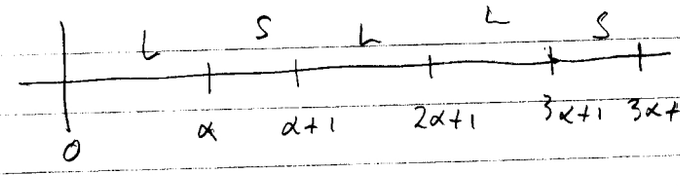
pound on table.
Sounds stupid!
Then play Pachibel canon

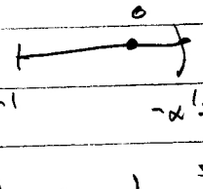


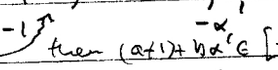
Fibonacci Music -

Ex: $\alpha = \frac{1+\sqrt{5}}{2} \simeq 1.618\ldots$ $\quad \alpha' = \frac{1-\sqrt{5}}{2} \simeq -0.618\ldots$

$$\overline{T} = \left\{ a+b\alpha \;;\; a, b \geq 0,\ a, b \in \mathbb{Z},\ a+b\alpha' \in [-1, -\alpha') \right\}$$





Facts: ① $\alpha T \subseteq T$. If $a+b\alpha'$ is here (Starting at $a+b\alpha$)  then $(a+1)+b\alpha' \in [-1, -\alpha')$ and $a+1+b\alpha$ is the next note!

② There are two types of intervals between notes:
L = long (length α)
S = short (length 1)
Multiplication by α sends an L to LS and an S to L:

③ Another way to produce the score:
Iterate the transformation L → LS, S → L

starting with the score L
L S
L S L
L S L L S
This produces arbitrarily long initial part of $\overline{T}$.

(3) Part 2 leads to an inflation rule for producing the score. Starting with L, we get the first two rules LS from $\alpha L = LS$. Iterate $L \to LS$, $S \to L$ to get arbitrarily long initial pieces of the score

L

LS

LSL

LS|LLS

(4) Harmony.

player 1	L	S	L	L S
player 2	αL	αS	αL	$\text{tempo}_2 = \frac{1}{\alpha}\text{tempo}_1$
player 3	$\alpha^2 L$	$\alpha^2 S$		$\text{tempo}_3 = \frac{1}{\alpha}\text{tempo}_2$

They are all playing the same eq. of long + short notes, at tempos decreasing geometrically by powers of α.

Play the music. Note

Toy	Art
⓪	Escher
	Pachabel Canon
Fibonacci music	?

Geometric Quasicrystals

W = lattice $\subseteq \mathbb{R}^n = H \oplus H^\perp$ orthogonal linear subspaces

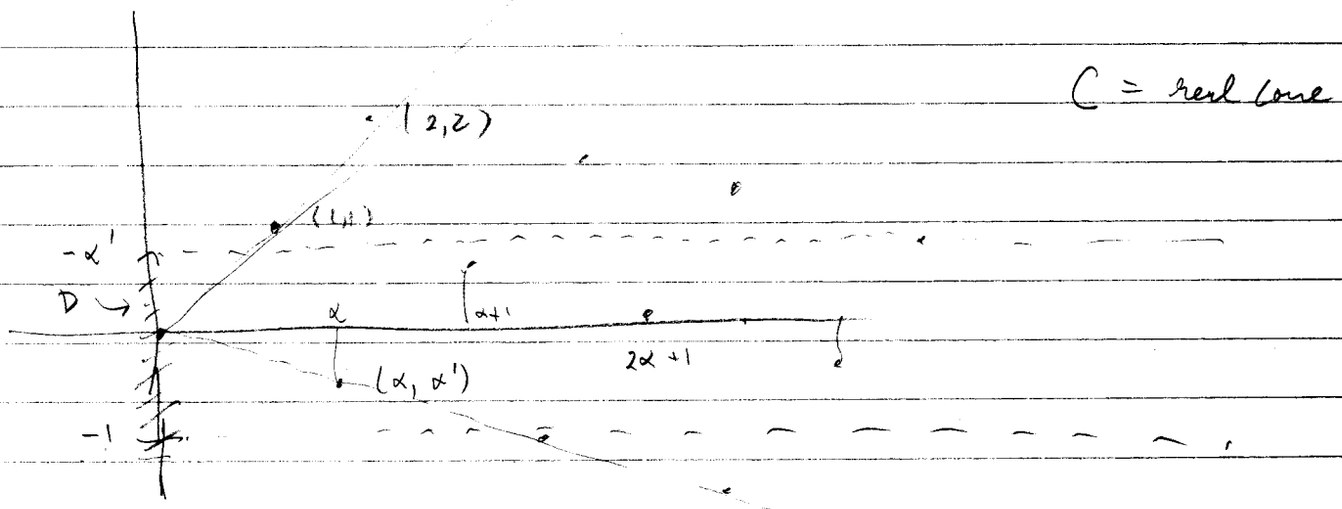
H ; $H^\perp$

C = real cone ; D = bounded.

$$T = \left\{ \mathrm{proj}_H(w) \,;\; w \in W \cap C \,,\; \mathrm{proj}_{H^\perp}(w) \in D \right\}.$$

This produces vertices of a quasicrystal.

$n = 2$ $\quad W = \{(a + b\alpha, a + b\alpha') \in \mathbb{R}^2\}$



: K arbitrary Galois ext $W \subseteq \mathcal{O}_K$ = integers in a tot. real K.

Another count:

T = starting points of long beats $\subseteq \{0, \ldots, n-2\}$.

T is 2-free: $i, j \in T \Rightarrow i = j$ or $|i-j| \geq 2$

2 free subsets of $\{0, \ldots, n-2\}$ $\longleftrightarrow$ ragas of length n.

Problem: A poor musician wants to play all the possible ragas in some sequence, but has just one sheet of paper. Can he order the ragas $T_1, T_2, \ldots, T_{F_n}$

so to go from one to the next, he only changes one L = ▭ to SS = ▭▭ or changes one SS = ▭▭ to L = ▭ ?

A Gray code order for subsets $T_1, \ldots, T_{F_n} \subseteq \{0, \ldots, n-2\}$

is one for which T_{i+1} differs from T_i by adding or deleting one element.

Fact: The musician can do it: There's a gray code order of the 2-free subsets of $\{0, \ldots, N\}$, $N = n-2$

Define: L_N = a Gray code listing of the 2 free subsets by

$L_0 = \phi$

$L_1 = \phi, \{0\}$

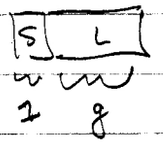
$L_N = \overline{L_{N-1}}, \; N \oplus L_{N-2}$

$\overline{L_{N-1}}$: reverse ordering of L_{N-1}

$N \oplus L_{N-2}$: add N to listing L_{N-2} in reverse order.

$H_N = \{N, N-3, N-6, \ldots\}$ = last elt of L_N = first of L_{N+1}.

What if long note is $g \geq 2$ times as long is the short one?



Ragas $\leftrightarrow$ g block free $T \subseteq \{0, \ldots, n-g\}$.
(so $i, j \in T \Rightarrow i = j$ or $|i-j| \geq g$).

raga $\leftrightarrow$ Starting points of long beats

$z_{n,g}$ = # ragas of length n = $z_{n-1,g}$ + $z_{n-g,g}$.

start with a short — start with a long

Thm (Savage, Wilf, C) There is an $N(g)$ depending on g so that if $N > N(g)$, there is no Gray code collection $B(g,N)$ of listing of the g-block free subsets of $\{0, \ldots, N\}$.

(So the musician can use (sheet of paper as before).

Ideas: Suppose there were a Gray code order

$T_1, T_2, T_3, \ldots$

$\#T_{i+1} = \#T_i \pm 1$. $\forall i$

$d(g,N) = \#\{T \in B(g,N) : \#T \text{ even}\} - \#\{T \in B(g,N) : \#T \text{ odd}\}$

$|d(g,N)| \leq 1$

Generating Functions ⇒

(*) $$d(g,N) = \sum_{i=1}^{g} A_{\theta_i} \theta_i^{-N} + A_0 \quad \text{if } N >> 0$$

where $\{\theta_i\}$ = roots of $x^g - x - 1 = 0$.

$A_0, A_{\theta_i} \in Q(\theta_1, \ldots, \theta_g)$ is algebraic.

and either ① there's a unique root $\theta \in \mathbb{R}$ so $|\theta| < |\theta_i|$ if $\theta_i \neq \theta$.

② there's a complex root $\theta \notin \mathbb{R}$ so $|\theta| = |\bar{\theta}| < |\theta_i|$ if $\theta_i \notin \{\theta, \bar{\theta}\}$.

③ the constants A_θ and $A_{\bar{\theta}}$ associated to θ in (*) in ① and ② are not 0.

Thus: In case ①: $d(g,N) = A_\theta \theta^{-N}$ + lower order terms $\to \pm\infty$ as $N \to \infty$

and $|d(g,N)| \leq 1$ is impossible if $N >> 0$

In case ②: Expect:

$$d(g,N) = A_\theta \theta^{-N} + A_{\bar{\theta}} \bar{\theta}^{-N} + \text{lower order}$$

$\to \infty$ as $N \to \infty$

But does this really happen.

Thm: Results of Baker + Feldman on linear forms in logs imply that $A_\theta \theta^{-N} + A_{\bar{\theta}} \bar{\theta}^{-N}$ is the main terms and $d(g,N) \to +\infty$ as $N \to \infty$