MATH 503: HOMEWORK #2B

DUE FRIDAY, FEB. 22, 5 P.M. YING ZONG'S MAILBOX

1. Products and Coproducts

1.1. In class we proved that if R is a division ring, then every left (or right) R-module is free, i.e. has a basis. Show that if R is a ring and there is a non-zero element $a \in R$ such that $Ra \neq R$, then there is a left R-module which is not free. Conclude that if every left R-module is free and every right R-module is free then R has to be a division ring.

1.2 Do problem 24 of section 10.3 of Dummit and Foote. This proves that the countably infinite product $\prod_{i=1}^{\infty} \mathbb{Z}$ of copies of $\mathbb{Z}$ is not a free $\mathbb{Z}$-module. You can use without proof the following facts about countable sets:

i. By definition, a set A is countable if it is either finite or there is a bijection between A and the positive integers $\mathbb{Z}_{>0}$. In other words, A is countable if it can be listed using either a finite set of integers or the elements of $\mathbb{Z}_{>0}$ as subscripts.

ii. The countable union of countable sets is countable. You can assume this, but it is not hard to prove if you think of finding a suitable listing scheme for the union.

iii. If $\{B_i\}_{i=1}^{\infty}$ is a collection of sets such that each B_i has at least two elements, the product set $\prod_{i=1}^{\infty} B_i$ is not countable. You can assume this, but it is not hard to prove using the following diagonal argument due to Cantor. If $\ell_1, \ell_2, \ldots$ were a countable listing of the elements of $\prod_{i=1}^{\infty} B_i$, and we think of each ℓ_i as a vector $(b_{i,1}, b_{i,2}, \ldots)$ of elements $b_{i,j} \in B_j$, then we can construct an element $b = (b_1, b_2, \ldots)$ which is not on the list by choosing for each j the element $b_j \in B_j$ to be different from $b_{j,j}$.

2. Vector spaces

2.1 Dummit and Foote give a proof on pages 410-411 of their book that if V is a vector space over a field F having a finite basis B, then any other basis for V over F is finite and has the same number of elements as B. This exercise gives a different (and arguably simpler) proof of this.

a. Show that it is enough to prove that if $\#B = \{b_1, b_2, \ldots, b_m\}$ has m distinct elements and $T = \{t_1, \ldots, t_{m+1}\}$ is any subset of V having $m+1$ elements, then T is not linearly independent. Check this statement when $m = 0$, i.e. when $V = \{0\}$!

b. Suppose $m > 0$. With the notation of part (a), show that the quotient F-vector space $V' = V/(Fb_1)$ has a basis with $m-1$ elements. Conclude by induction on m that there is an equality

$$r_1 t_1 + r_2 t_2 + \cdots r_m t_m = r b_1$$

in V for some $r, r_1, \ldots, r_m \in F$ such that $r_1, \ldots, r_m$ are not all 0. If $r \neq 0$ argue that we are done, so that we can suppose $r \neq 0$.

c. After relabeling the elements of T, we can assume that in the relation produced in part (b), one has $r_1 \neq 0$. Using V' and induction on m again, show there is an equality

$$r'_2 t_2 + r'_3 t_3 + \cdots r'_m t_m + r'_{m+1} t_{m+1} = r' b_1$$

in V for some $r', r'_2, r'_3, \ldots, r'_{m+1} \in F$ in which we can assume $r' \neq 0$. Show there is a linear combination of the equalities produced in parts (b) and (c) which shows T is linearly dependent.

2.2. Recall that two sets have the same cardinality if there is a set bijection between them. This exercise will show that any two bases B and B' of a given vector space V over a field F have the same cardinality, generalizing exercise 2.1. (We'll assume some fact about cardinalities proved in part 3 of Appendix 2 of Lang's Algebra book, though.) The cardinality of a basis for V over F is called the dimension $\dim_F(V)$ of V over F.

a. By exercise 2.1 we can assume that B and B' are infinite. Let $Z(B)$ be the set of all finite subsets of B. For $z = \{b_1, \ldots, b_n\} \in Z(B)$, let $V(z)$ be the F-vector subspace of V having z as a basis. Let $W(B)$ be the set $\{V(z) : z \in Z(B)\}$. Show that the map $f : Z(B) \to W(B)$ defined by $z \to V(z)$ is a bijection.

b. Let $Z(B')$ be the set of all finite subsets of B'. Show that if $z' = \{b'_1, \ldots, b'_m\} \in Z(B')$, the F-vector space $V(z')$ spanned by z' is contained in $V(z)$ for some $z \in Z(B)$. Prove that there is a unique smallest subset $h(z') \in Z(B)$ such that $V(z') \subset V(h(z'))$. (Hint: Every element of B' is a finite F-linear combination of element of B.)

c. Show that the map $h : Z(B') \to Z(B)$ defined by $z' \to h(z')$ in part (b) has finite fibers, in the sense that there are only finitely many z' for which $h(z')$ is a specified element of $Z(B)$. You will need to use the fact that finite dimensional vector spaces have well defined dimensions.

d. If S and T are sets and there is an injection $S \to T$ we say $\mathrm{card}(S) \leq \mathrm{card}(T)$; if there is a bijection $S \to T$ then say $\mathrm{card}(S) = \mathrm{card}(T)$. Complete the proof that B and B' have the same cardinality using without proof the following facts from Appendix 2 of Lang's Algebra book.

i. Lang proves in Theorem 3.3 on Appendix 2 that if I is an infinite set and D is a countable set then $\mathrm{card}(I \times D) = \mathrm{card}(I)$. Use this to show that if $\{A_i\}_{i \in I}$ is a set of finite sets indexed by an infinite set I then $\mathrm{card}(\cup_{i \in I} A_i) \leq \mathrm{card}(I)$.

ii. Take I to be the image of the map $h : Z(B') \to Z(B)$ in part (c) of this problem. Show that $\mathrm{card}(Z(B')) \leq \mathrm{card}(Z(B))$.

iii. Lang proves in Theorem 3.6 of Appendix 2 that if B' is infinite then $\mathrm{card}(Z(B')) = \mathrm{card}(B')$. Conclude that $\mathrm{card}(B') \leq \mathrm{card}(B)$. By interchanging the roles of B and B' we get also that $\mathrm{card}(B) \leq \mathrm{card}(B')$.

iv. The Schroeder-Bernstein Theorem says that if A and C are sets and $\mathrm{card}(A) \leq \mathrm{card}(C)$ and $\mathrm{card}(C) \leq \mathrm{card}(A)$ then $\mathrm{card}(A) = \mathrm{card}(C)$. This is a highly non-trivial assertion! See Theorem 3.1 of Lang's appendix for a proof. Assuming this fact, complete the proof that B and B' have the same cardinality.