MATH 503: HOMEWORK #4B

DUE MONDAY, APRIL 7 BY 5 P.M. IN YING ZONG'S MAILBOX

1. Categories and functors

1.1 Do exercise 2 of §1 of Appendix II of Dummit and Foote, and then do exercise 2 of §2 of Appendix II.

1.2 Do exercise 3 of §1 of Appendix II Dummit and Foote. (For the definition of full and faithful functors, see the last paragraph §1 of Appendix II.)

2. Projective, injective and flat modules

2.1 Do exercise 1(a,b) of §10.5 of Dummit and Foote.

2.2 Do exercise 3 of §10.5 of Dummit and Foote.

2.3 Let R be a ring. An exact sequence

$$N^\bullet : 0 \longrightarrow N_1 \xrightarrow{\pi_1} N_2 \xrightarrow{\pi_2} N_3 \longrightarrow 0$$

of R-modules is said to split if there is an R-module homomorphism $s : N_3 \to N_2$ such that $\pi_2 \circ s : N_3 \to N_3$ is the identity homomorphism. In this case, the R-module homomorphism $N_1 \oplus N_3 \to N_2$ defined by $n_1 \oplus n_2 \to \pi_1(n_1) + s(n_3)$ is an isomorphism (see the discussion after Proposition 24 in §10.5 of Dummit and Foote). It is shown in Propositions 30 and 34 of §10.5 of Dummit and Foote that $N^\bullet$ splits if either N_3 is projective or N_1 is injective. Use this to show that the following conditions on R are equivalent:

i. Every R-module is projective.

ii. Every R-module is injective.

2.4 Do exercise 26 on §10.5 of Dummit and Foote. (You can quote the result of exercise 25 on which this depends, without writing up a proof for exercise 25). Explain how to use this result to give an example of a flat $\mathbf{Z}$-module which is not projective.

2.5 Do exercise 3 of §17.1 of Dummit and Foote.