MATH 621: HOMEWORK #1

1. Quaternion and Dihedral extensions of $\mathbb{Q}$.

This problem has to do with constructing degree 8 quaternion and dihedral extensions using class field theory.

1. Suppose H is a subgroup of a finite group G. The transfer homomorphism

$$\mathrm{Ver}_G^H : G^{ab} \to H^{ab}$$

between the maximal abelian quotients of G and H is defined in the following way. Let T be a set of representatives for the right cosets of H in G, so that $H\backslash G = \{Ht : t \in T\}$. If $g \in G$ and $t \in T$, then $tg = h_{g,t}t'$ for some $t' \in T$ and $h_{g,t} \in H$. Define

$$\mathrm{Ver}_G^H(\overline{g}) = \overline{h} \quad \text{when} \quad h = \prod_{t \in T} h_{g,t}$$

where $\overline{g}$ (resp. $\overline{h}$) is the image of g in G^{ab} (resp. the image of h in H^{ab}). Show that if H is cyclic of order 8 and G is a dihedral (resp. quaternion) group of order 8, then Ver_G^H is trivial if G is dihedral, and otherwise Ver_G^H is the unique non-trivial homomorphism which has kernel the image of H in G^{ab}.

2. Let K be a global field, with idele class group $C_K = J_K/K^*$. Show that all dihedral and quaternion extensions of K arise from the following construction. Let L/K be a quadratic normal extension, and let $\epsilon_L : C_K \to \{\pm 1\}$ be the unique surjective homomorphism corresponding to L via class field theory. Write $\mathrm{Gal}(L/K) = \{e, \sigma\}$, with σ of order 2. Let $\mu_4 = \{\pm 1, \pm\sqrt{-1}\}$ be the group of fourth roots of unity in $\mathbb{C}^*$. A surjective homomorphism $\chi : C_L \to \mu_4$ is of dihedral (resp. quaternion) type if:

a. $\chi^\sigma = \chi^{-1}$ when $\chi^\sigma : C_L \to \mu_4$ is defined by $\chi^\sigma(j) = \chi(\sigma(j))$ for $j \in C_L$

b. The restriction $\chi|_{C_K}$ of χ to C_K via the map $C_K \to C_L$ induced by including K into L is trivial (in the dihedral case) or the character ϵ_L (in the quaternion case).

Let N be the extension of L which corresponds to the kernel of χ via class field theory over L. Show that N/K is a dihedral (resp. quaternion) extension of degree 8 if χ is of dihedral (resp. quaternion) type, and that all such extensions arise from this construction as L ranges over the quadratic Galois extensions of K. Which pairs (L, χ) give rise to the same N?

3. The character $\chi : C_L = J_L/L^* \to \mu_4$ then has local components $\chi_v : L_v^* \to \mu_4$ for each place v of L defined by $\chi_v(j_v) = \chi(\iota_v(j_v))$ when $\iota_v : L_v^* \to C_L$ results from the inclusion of L_v into J_L at the place v followed by the projection $J_L \to C_L/L^*$.

a. Suppose K is a number field and that K and L have class number 1. Show that there are exact sequences

$$1 \to O_L^* \to \prod_v O_v^* \to C_L \to 1 \quad \text{and} \quad 1 \to O_K^* \to \prod_w O_w^* \to C_K \to 1 \tag{1.1}$$

where v and w range over all places of L and K, respectively, including the archimedean places. Conclude from this that to specify a finite order continuous homomorphism

$\chi : C_L \to \mathbb{C}^*$ it is necessary and sufficient to specify continuous local characters $\chi'_v : O_v^* \to \mathbb{C}^*$ which are trivial for almost all v such that $\prod_v \chi'_v$ vanishes on O_L^*.

b. With the notations of problem (3a), what conditions on the restrictions χ'_v are equivalent to χ being of dihedral or quaternion type? (Note that by the same reasoning, the character $\epsilon : C_K \to \{\pm 1\}$ is determined by its restrictions to the multiplicative groups O_w^* of all places w of K, and that each such O_w^* embeds naturally into the product of the O_v^* associated to v over w in L.)

c. Suppose $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{5})$. Show that there is a quaternion character $\chi : C_L \to \mu_4$ such that the $\chi'_v = \chi_v|O_v^*$ have the following properties. The character χ'_v is trivial unless v is the unique place v_5 over 5 or one of the two first degree places v_{41} and v'_{41} over 41. The order of χ'_v is 2 if $v = v_5$ and 4 if $v = v_{41}$ or $v = v'_{41}$. Finally, when we use the natural inclusion $K = \mathbb{Q} \to L$ to identify both $O_{v_{41}}$ and $O_{v'_{41}}$ with $\mathbb{Z}_{41}$, the characters $\chi'_{v_{41}}$ and $\chi'_{v'_{41}}$ are inverses of each other when we view them both as characters of $\mathbb{Z}_{41}^*$.

2. The Carlitz module and class field theory

Homework #3 of last semester included some problems about abelian extensions of $L = \mathbb{F}_p(t)$ when p is a prime which are constructed using the Carlitz module. To recall this construction, let $A = \mathbb{F}_p[t]$. One has a ring homomorphism $\psi : A \to L\{\tau\}$ sending t to $t + \tau$, where $L\{\tau\}$ is the twisted polynomial ring for which $\tau\beta = \beta^p\tau$ for $\beta \in L$. Then $L\{\tau\}$ acts on an algebraic closure $\overline{L}$ of L by letting $\beta \in L$ act by multiplication by β, and by letting τ send $\alpha \in \overline{L}$ to $\tau(\alpha) = \alpha^p$. If $\pi(t) \in A$ is not 0, define the $\pi(t)$-torsion subgroup of $\overline{L}$ by

$$\mu_{\pi(t)} = \{\alpha \in \overline{L} : \psi(\pi(t))(\alpha) = 0\}$$

Suppose $\pi(t) \in A = \mathbb{F}_p[t]$ is monic of degree $d \geq 1$ in t. Homework # 3 of last semester showed the following facts:

1. $\mu_{\pi(t)}$ is the set of all roots of a separable polynomial of degree p^d, and $\mu_{\pi(t)}$ is an additive group.

2. There is an action of the ring $A/\pi(t)A$ on $\mu_{\pi(t)}$ induced by letting the class of $h(t) \in A$ send $\alpha \in \mu_{\pi(t)}$ to $\psi(h(t))(\alpha)$. This makes $\mu_{\pi(t)}$ into a free rank one module for $A/\pi(t)A$.

3. Let $L(\mu_{\pi(t)}) = N$ be the field obtained by adjoining to L all elements of $\mu_{\pi(t)}$. Suppose $\pi(t)$ is a monic irreducible polynomial of degree d. Let $\alpha \in \mu_{\pi(t)}$ be a generator for $\mu_{\pi(t)}$ as a free rank one module for the field $A/A\pi(t)$. The integral closure of $B = \mathbb{F}_p[t]$ in the field $L(\mu_{\pi(t)})$ obtained by adjoining to L all elements of $\mu_{\pi(t)}$ is the ring $B[\alpha]$ generated by B and α. (The proof is analogous to showing that $\mathbb{Z}[\zeta_p]$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\zeta_p)$.)

For simplicity we now assume that as in # 3 above, $\pi(t)$ is a monic irreducible of degree d. Here are some problems about relating this construction to class field theory.

a. Use #2 and #3 above to show that N is an abelian extension of L with Galois group equal to the unit group $(A/\pi(t)A)^*$ of the ring $A/\pi(t)A$. Show that N is totally ramified over the place of L associated to $\pi(t)$. (Hint: One can follow the pattern of the proof that $\mathbb{Q}(\zeta_p)$ is an abelian extension of $\mathbb{Q}$ with Galois group $(\mathbb{Z}/p)^*$.)

b. Suppose that $f(t)$ is a monic irreducible polynomial in $B = \mathbb{F}_p[t]$ which is different from $\pi(t)$. Show that the place v of L determined by $f(t)$ is unramified in L. Then show that if w is any place of N over L, the Frobenius automorphism $\mathrm{Frob}(w) \in G = \mathrm{Gal}(N/L)$ associated to w is the image of $f(t)$ in $(A/\pi(t)A)^*$ when we identify $(A/\pi(t)A)^*$ with G as in part (a) above. (Hint: To see what is going on here, write down explicitly the case in which $\pi(t) = t$ and $f(t) = t - \beta$ for some non-zero $\beta \in \mathbb{F}_p$.)

c. Conclude from part (b) that N/L is unramified outside the place v_0 of $L = \mathbb{F}_p(t)$ determined by $\pi(t)$ and the place v_∞ such that $\mathrm{ord}_{v_\infty}(g(t)) = -\deg(g(t))$ for $g(t) \in \mathbb{F}_p[t]$. In class we shows that the degree map on the ideles J_L of L gives an exact sequence

$$1 \to J_L^0 \to J_L \to \mathbb{Z} \to 0 \tag{2.2}$$

where

$$J_L^0 = L^* \times \left(\prod_{v \neq v_\infty} O_v^* \right) \times (1 + t^{-1} O_{v_\infty}^*) \tag{2.3}$$

and t^{-1} is a uniformizer in O_{v_∞}. Using this description and part (b), write down the Artin map

$$\psi_{N/L} : C_L = J_L/L^* \to \mathrm{Gal}(N/L) = (A/\pi(t)A)^* \tag{2.4}$$

Is N/L ramified over v_∞? (Hint: First consider the restriction of $\psi_{N/L}$ to J_L^0/L^*, and use the fact that v_0 is totally ramified in N.)