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Topological properties of Eschenburg spaces and 3-Sasakian manifolds

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Abstract We examine topological properties of the seven-dimensional positively curved Eschenburg biquotients and find many examples which are homeomorphic but not diffeomorphic. A special subfamily of these manifolds also carries a 3-Sasakian metric. Among these we construct a pair of 3-Sasakian spaces which are diffeomorphic to each other, thus giving rise to the first example of a manifold which carries two non-isometric 3-Sasakian metrics.

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Riemannian manifolds with positive sectional curvature have been a frequent topic of global Riemannian geometry for over 40 years. Nevertheless, there are relatively few known examples of such manifolds. The purpose of this article is to study the topological properties of some of these examples, the so-called Eschenburg spaces, in detail.

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In addition to positively curved metrics, some Eschenburg spaces also carry another special geometric structure, namely a 3-Sasakian metric, i.e., a metric whose Euclidean cone is Hyperkähler [5]. 3-Sasakian spaces are interesting since they are Einstein manifolds and are connected to several other geometries: they admit an almost free, isometric action by $\mathrm{SU}(2)$ whose quotient is a quaternionic Kähler orbifold. The twistor space of this orbifold, which can be viewed as an S^1-quotient of the 3-Sasakian manifold, carries a natural Kähler–Einstein orbifold metric with positive scalar curvature.

3-Sasakian structures are rare and rigid, in fact the moduli space of such metrics on a fixed manifold consists of at most isolated points. This motivated Boyer and Galicki [5][Question 9.9, p. 52] to pose the question whether a manifold can admit more than one 3-Sasakian structure. Natural candidates for such examples are the 3-Sasakian metrics discovered in [4]. They are defined on the Eschenburg biquotients $E_{a,b,c} = \mathrm{diag}(z^a, z^b, z^c)\backslash \mathrm{SU}(3)/\,\mathrm{diag}(z^{a+b+c}, 1, 1)$, where a, b, c are positive, pairwise relatively prime integers. The simplest topological invariant of these spaces is the order of the fourth cohomology group, which is a finite cyclic group of order $r = ab + ac + bc$. By studying further topological invariants of these manifolds we show:

Theorem A *For $r \leq 10^7$, there is a unique pair of 3-Sasakian Eschenburg spaces $E_{a,b,c}$ which are diffeomorphic to each other, but not isometric. It is given by $(a, b, c) = (2279, 1603, 384)$ and $(2528, 939, 799)$ with $r = 5{,}143{,}925$.*

The two 3-Sasakian metrics are non-isometric since the isometric action by $\mathrm{SU}(2)$ has cyclic isotropy groups of order $a+b$, $a+c$ and $b+c$. In [5] they also asked whether two 3-Sasakian manifolds can be homeomorphic to each other but not diffeomorphic. This happens frequently among the 3-Sasakian Eschenburg spaces. There are 139 such pairs for $r \leq 10^7$, the first one of which is given by $(a, b, c) = (171, 164, 1)$ and $(223, 60, 53)$ for $r = 28{,}379$. See Table 5 for the next 4 such pairs.

The manifolds $E_{a,b,c}$ also carry a metric of positive sectional curvature, although the 3-Sasakian metric never has positive curvature. They are special cases of the more general family of Eschenburg spaces given by $E_{k,l} = \mathrm{diag}(z^{k_1}, z^{k_2}, z^{k_3})\backslash \mathrm{SU}(3)/\,\mathrm{diag}(z^{l_1}, z^{l_2}, z^{l_3})$. In this article we also examine the topology of the positively curved spaces among this more general class of Eschenburg spaces. They contain in particular the homogeneous Aloff–Wallach spaces $\mathrm{SU}(3)/\,\mathrm{diag}(z^p, z^q, \bar{z}^{p+q})$, [1], which Kreck–Stolz [12] classified up to homeomorphism and diffeomorphism. They were thus able to construct the first examples of positively curved Riemannian manifolds which are homeomorphic but not diffeomorphic. For this purpose they introduced three invariants for seven-dimensional manifolds with the same cohomology ring as $W_{p,q}$ or $E_{k,l}$, which are generalizations of the classical Eells–Kuiper invariant. They are computed using a bounding eight-dimensional manifold and detect both the homeomorphism and diffeomorphism type. In case of the Aloff–Wallach spaces, which can be viewed as circle bundles over the homogeneous flag manifold, the bounding manifold is simply the corresponding disc bundle. Another special case, namely the circle bundles over the inhomogeneous six-dimensional flag manifold, were studied in [2,3] although in this case there are not even any homeomorphic pairs, see [16]. The invariants for the general Eschenburg family, for which it is more difficult to find a bounding eight-manifold, were computed by Kruggel [15]. We use his formulas to study the topology of $E_{k,l}$.

The fourth cohomology group of $E_{k,l}$ is a finite cyclic group of order $r = |k_1k_2 + k_1k_3 + k_2k_3 - (l_1l_2 + l_1l_3 + l_2l_3)|$, and we show that for a given value of r there are only finitely many positively curved Eschenburg spaces.

Theorem B *For $r \leq 8{,}000$, there is a unique pair of positively curved Eschenburg spaces $E_{k,l}$ which are homeomorphic to each other, but not diffeomorphic, given by* $(k_1, k_2, k_3 \mid l_1, l_2, l_3) = (79, 49, -50 \mid 0, 46, 32)$ *and* $(75, 54, -51 \mid 0, 46, 32)$ *with* $r = 4{,}001$.

There are 69 pairs of this type for $r \leq 50,000$, the first 5 are listed in Table 2. Among these 69 there are also 4 pairs which are diffeomorphic to each other, see Table 3. For one of these pairs, one positively curved metric is cohomogeneity four, whereas the other is cohomogeneity two. In the case of the Aloff-Wallach examples in [12] the integer parameters must be significantly larger. They find 11 homeomorphic pairs and 3 diffeomorphic pairs for $r < 10^{17}$.

We point out that in contrast to the above results, it happens much more frequently that Eschenburg spaces which are not positively curved are homeomorphic or diffeomorphic to each other. For example, the cohomogeneity two Eschenburg spaces $(k_1, k_2, k_3 \mid l_1, l_2, l_3) = (-71, 97, 265 \mid 0, 0, 291)$ and $(-215, 397, 469 \mid 0, 0, 651)$ are diffeomorphic to the positively curved Aloff-Wallach space $(1, 1, -2 \mid 0, 0, 0)$.

To prove these theorems we use the Kreck–Stolz invariants, as described in [15]. In contrast to the case of the circle bundles in [12] and [2], where the formulas for the invariants are fourth degree polynomials, the formulas in [15] are quite complicated and involve several number theoretic sums, with values in $\mathbb{Q}/\mathbb{Z}$. In order to compute these invariants on a computer, one needs to control the denominators, see Theorem 3.1.

Our general strategy is to first find pairs of Eschenburg respectively 3-Sasakian manifolds where the following basic invariants match: the integer r, the linking form, and the first Pontrjagin class. The latter two invariants can be interpreted as integers modulo r and have simple algebraic expressions. Secondly, for this surprisingly small list of pairs, we compute the Kreck–Stolz invariants. We point out that Kruggel [15] proved his formulas only under the assumption that his condition (C) holds. However, we show that this condition is not always satisfied, even for positively curved Eschenburg spaces; and thus there is in fact no general formula for the Kreck–Stolz invariants for all Eschenburg spaces. Fortunately, those pairs of spaces with matching basic invariants, which we find, all satisfy condition (C).

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that carries two non-isometric 3-Sasakian metrics, is given by

$$(a, b, c) = (4219, 2657, 217) \quad \text{and}$$
$$(a, b, c) = (4637, 1669, 787) \text{ with } r = 12{,}701, 975.$$

1 Eschenburg spaces

A biquotient is a generalization of a homogeneous space where $H \subset G \times G$ acts on G via $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$. The action is free if and only if h_1 is never conjugate to h_2, in which case the quotient is a manifold denoted by $G//H$. We will also use the notation $\phi_1(h)\backslash G/\phi_2(h)$ where the inclusion $H \subset G \times G$ is given by $(\phi_1(h), \phi_2(h))$.

The Eschenburg spaces are an infinite family of seven dimensional manifolds containing a subfamily that admits a metric a positive sectional curvature. They were introduced by Eschenburg [7], and they can be described as biquotients of SU(3).

Let $k := (k_1, k_2, k_3), l := (l_1, l_2, l_3) \in \mathbb{Z}^3$ be two triples of integers such that $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$. We can then define a two-sided action of $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ on SU(3) whose quotient we denote by $E_{k,l}$:

$$E_{k,l} := \operatorname{diag}(z^{k_1}, z^{k_2}, z^{k_3})\backslash \operatorname{SU}(3)/\operatorname{diag}(z^{l_1}, z^{l_2}, z^{l_3}), k_1 + k_2 + k_3 = l_1 + l_2 + l_3.$$

The action is free if and only if $\operatorname{diag}(z^{k_1}, z^{k_2}, z^{k_3})$ is not conjugate to $\operatorname{diag}(z^{l_1}, z^{l_2}, z^{l_3})$ which translates into the following conditions, which must all be satisfied:

$$\begin{aligned} &\gcd(k_1 - l_1, k_2 - l_2) = 1 \quad \gcd(k_1 - l_2, k_2 - l_1) = 1 \\ &\gcd(k_1 - l_1, k_2 - l_3) = 1 \quad \gcd(k_1 - l_2, k_2 - l_3) = 1 \\ &\gcd(k_1 - l_3, k_2 - l_1) = 1 \quad \gcd(k_1 - l_3, k_2 - l_2) = 1 \end{aligned} \tag{1.1}$$

In [7] it is also shown that the spaces $E_{k,l}$, equipped with a metric induced by a certain left invariant metric on SU(3), have positive sectional curvature if and only if:

$$k_i \notin [\min(l_1, l_2, l_3), \max(l_1, l_2, l_3)] \quad \text{for all } i = 1, 2, 3. \tag{1.2}$$

Remark 1.3 One of the difficulties of dealing with these spaces, is that they do not have a unique representation. One can easily change the integers such that the S^1-actions are equivalent to each other, and hence the quotient manifolds are diffeomorphic:

- We can use any permutation of the entries in k since an element of the Weyl group of SU(3) acting on the left will produce an equivalence of the corresponding actions. Similarly, we can use any permutation of the entries of l.
- We can switch all entries in k with the entries in l, if we replace the left-invariant metric with a right invariant one, since the inversion on SU(3) induces an isometry.
- Simultaneously changing the signs of all entries in k and l is obtained by pre-composing the action with $z \mapsto \bar{z}$. Note though that in this case, the operation changes the orientation of $E_{k,l}$.

– Adding an integer to all entries in k and l, i.e. replacing k_i and l_i by $k_i + n$ and $l_i + n$ for $n \in \mathbb{Z}$, induces the same action of S^1.

In the case of positively curved Eschenburg spaces, we use such changes of the group action to obtain a unique representation:

Lemma 1.4 *Each positively curved Eschenburg space $E_{k,l}$ has the following unique representation*

$$\begin{aligned} k &= (k_1, k_2, l_1 + l_2 - k_1 - k_2) \\ l &= (l_1, l_2, 0) \quad \text{with } k_1 \geq k_2 > l_1 \geq l_2 \geq 0. \end{aligned} \tag{1.5}$$

Proof Recall that the k_i and l_i have to satisfy the positive curvature condition (1.2). If necessary change the signs of all k_i and l_i to ensure that both k_1 and k_2 are on the right of the interval $[\min(l_1, l_2, l_3), \max(l_1, l_2, l_3)]$. We can assume that $k_1 \geq k_2$ by changing the order of k_1 and k_2 if necessary. Now subtract $\min(l_1, l_2, l_3)$ and after possibly changing the order of l_1 and l_2 we can assume that $l_1 \geq l_2 \geq l_3 = 0$. Hence we obtain that $k_1 \geq k_2 > l_1 \geq l_2 \geq l_3 = 0 > k_3 = l_1 + l_2 - k_1 - k_2$. □

Using the Serre spectral sequence as in [9], or by following the methods developed in [17], one obtains the cohomology ring with integer coefficients:

$$\begin{aligned} &H^1(E_{k,l}) = 0,\ H^2(E_{k,l}) = \mathbb{Z} \quad \text{generated by } u, \\ &H^3(E_{k,l}) = 0,\ H^4(E_{k,l}) = \mathbb{Z}_{|r|} \quad \text{generated by } u^2, \quad \text{with } r = \sigma_2(k) - \sigma_2(l), \end{aligned} \tag{1.6}$$

where $\sigma_i(k) := \sigma_i(k_1, k_2, k_3)$ is the ith elementary symmetric function. Moreover, by studying the cell structure of the Eschenburg spaces as in [14, Remark 1.3], one proves that r is always an odd number.

In order to generate a complete list of all positively curved Eschenburg spaces the following proposition will be important:

Proposition 1.7 *For each odd $r \in \mathbb{Z}$ there are only finitely many positively curved Eschenburg spaces $E_{k,l}$ with $H^4(E_{k,l}) = \mathbb{Z}_{|r|}$.*

Proof Assume that $E_{k,l}$ is represented as in Lemma (1.4). Then we obtain that

$$\begin{aligned} r &= k_1k_2 + k_1k_3 + k_2k_3 - l_1l_2 \quad \text{with } k_3 = l_1 + l_2 - k_1 - k_2 \\ &= -[k_1(k_1 - l_1) + (k_2 - l_2)(k_1 + k_2 - l_1)] \quad \text{with } k_1 \geq k_2 > l_1 \geq l_2 \geq 0\,. \end{aligned}$$

Note that $k_1 > 0$, $k_1 - l_1 > 0$, $k_2 - l_2 > 0$ and $k_1 + k_2 - l_1 > 0$ and hence $r < 0$. If we fix a positive odd integer $N \in \mathbb{Z}^+$ with $N = -[k_1(k_1 - l_1) + (k_2 - l_2)\,(k_1 + k_2 - l_1)]$, then the above conditions imply $k_1, k_2, l_1, l_2 \in [1, N]$. Hence there are only finitely many choices for k_i, l_i as claimed. □

There are various interesting subfamilies of the Eschenburg spaces that have appeared in different contexts, see for example [2–4,6,12,16]. We use [20] for a systematic description of these subfamilies.

(1) Cohomogeneity one Eschenburg spaces

A group action of G on a manifold M is said to be of cohomogeneity one if the orbit space M/G is one-dimensional. For $E_{k,l}$ with $k_1 = k_2$ and $l_1 = l_2$ the group $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ acting on SU(3) on the left and on the right, clearly commutes with the S^1-action and induces a cohomogeneity one action on $E_{k,l}$. Using a change in the group action as in (1.3), we can rewrite these cohomogeneity one Eschenburg spaces as follows.

$$E_a = \mathrm{diag}\left(z^a, z, z\right) \backslash \mathrm{SU}(3) / \mathrm{diag}\left(z^{a+2}, 1, 1\right).$$

The action is free for all $a \in \mathbb{Z}$. Since $E_a = E_{-a-1}$, again via changes as in (1.3), we can assume $a \geq 0$. It follows that E_a has positive curvature for all $a \neq 0$. Note that in this case $r = 2a + 1$ and we obtain exactly one positively curved cohomogeneity one space for each odd r.

(2) Cohomogeneity two Eschenburg spaces

If two of the integers in k or in l are equal, we obtain an action of $G = \mathrm{SU}(2) \times T^2$ on SU(3) commuting with the S^1 action such that the orbit space $E_{k,l}/G$ is two dimensional. There are two families of Eschenburg spaces of this type.

(2^+) $k_1 = k_2$. We can rewrite these particular cohomogeneity two Eschenburg spaces as follows.

$$E_{a,b,c} = \mathrm{diag}\left(z^a, z^b, z^c\right) \backslash \mathrm{SU}(3) / \mathrm{diag}\left(z^{a+b+c}, 1, 1\right)$$

The action is free if a, b, c are pairwise relatively prime integers and $E_{a,b,c}$ has positive curvature if and only if $a \geq b \geq c > 0$. We can assume $a > b > c > 0$ since it is otherwise an Eschenburg space of cohomogeneity one. This subfamily also includes the circle bundles over the inhomogeneous flag manifold given by $a = b + c$, see [10], and their topological properties were studied in [2,3,16]. As mentioned in the introduction, they also admit a second metric which is 3-Sasakian and we will study this subfamily to prove Theorem A.

(2^-) $l_1 = l_2$. These spaces can be rewritten again as

$$\mathrm{diag}\left(z^a, z^b, z^c\right) \backslash \mathrm{SU}(3) / \mathrm{diag}\left(z^{a+b+c}, 1, 1\right)$$

with a, b, c pairwise relatively prime. But in this case they have positive curvature if and only if $a \geq b > 0$ and $c < -a$, and $a > b$ if they are not of cohomogeneity one. If $a + b + c = 0$ we recover the Aloff–Wallach spaces $W_{a,b} = \mathrm{SU}(3)/\,\mathrm{diag}(z^a, z^b, z^{-a-b})$.

(3) Cohomogeneity four Eschenburg spaces.

In the general case $G = T^3$ acts isometrically on $E_{k,l}$ and M/G is four dimensional. In our normalization we can assume $k_1 > k_2 > l_1 > l_2 \geq 0$.

In [11] it is shown that the above group actions cannot be extended to an isometric action with smaller cohomogeneity since the groups agree with the identity component of the isometry group.

2 Topological invariants

The topological invariants we use are the order of the fourth cohomology group $r := |r(k,l)| = |\sigma_2(k) - \sigma_2(l)|$, the self-linking number, the first Pontrjagin class, and the Kreck–Stolz invariants. All of these invariants were computed for most of the Eschenburg spaces in [15]. The results can be summarized as follows.

(1) The self-linking number of a class $u^2 \in H^4(E_{k,l})$ is given by

$$\mathrm{Lk}(k,l) := \mathrm{Lk}\left(E_{k,l}\right) = \mathrm{Lk}\left(u^2, u^2\right) = -\frac{s^{-1}(k,l)}{r(k,l)} \in \mathbb{Q}/\mathbb{Z}$$

where $s(k,l) := \sigma_3(k) - \sigma_3(l)$ and $s^{-1}(k,l)$ is the multiplicative inverse of $s(k,l)$ in $\mathbb{Z}_{|r(k,l)|}$. Notice that $\mathrm{Lk}(k,l)$ is uniquely determined by $s(k,l)$ up to sign (which will effect the orientation). In our normalization we always have $r(k,l) < 0$ and hence in our tables we will use $s(k,l) \mod |r(k,l)|$.

(2) The first Pontrjagin class, as an element in $H^4(E_{k,l}) \cong \mathbb{Z}_{|r(k,l)|}$, is given by

$$p_1(k,l) := p_1(E_{k,l}) = \left[2\,\sigma_1(k)^2 - 6\,\sigma_2(k)\right] \cdot u^2 \quad \in \mathbb{Z}_{|r(k,l)|}\,.$$

Note that the roles of k and l are interchangeable since $\sigma_1(k) = \sigma_1(l)$ and $\sigma_2(k) \equiv \sigma_2(l) \mod |r(k,l)|$. The second Stiefel–Whitney class vanishes for all $E_{k,l}$.

(3) The Kreck–Stolz invariants are based on the Eells–Kuiper μ-invariant and are defined as linear combinations of relative characteristic numbers of appropriate bounding manifolds. They were introduced in [12] and calculated for most of the Eschenburg spaces in [15]. One first constructs a cobordism of $\mathrm{SU}(3)$ and extends the S^1-action to this cobordism. However, in general this extended circle action is not free anymore and in order to make the action almost free, Kruggel introduced the following condition (C).

We say that condition (C) holds if and only if the matrix $A_{i,j} = (k_i - l_j)$ contains at least one row or one column whose entries are pairwise relatively prime. In [15] it was indicated that condition (C) might always be satisfied. Unfortunately, this is not the case. There are many Eschenburg spaces, even positively curved ones, which do not satisfy condition (C). For example for $r < 5{,}000$ there are 54 positively curved Eschenburg spaces for which condition (C) fails. The positively curved Eschenburg space with smallest $|r(k,l)|$ where this occurs is given by $(k_1, k_2, k_3 \mid l_1, l_2, l_3) = (35, 21, -34, \mid 12, 10, 0)$, with:

$$A = \begin{pmatrix} 23 & 5^2 & 5 \cdot 7 \\ 3^2 & 11 & 3 \cdot 7 \\ -2 \cdot 23 & -2^2 \cdot 11 & -2^2 \cdot 17 \end{pmatrix}$$

Determining the homeomorphism type and diffeomorphism type of Eschenburg spaces that do not satisfy condition (C) remains an open problem. Notice though that in the subclass of cohomogeneity two Eschenburg spaces, and in particular for 3-Sasakian manifolds, condition (C) always holds since in that case the first column

(a, b, c) consists of pairwise relatively prime integers. Fortunately, in the proof of Theorems B this problem does not arise either, as explained in Sect. 4

Assuming that condition (C) holds for the jth column, there are at most three exceptional orbits for the S^1 action on the cobordism of SU(3) with isotropy groups $\mathbb{Z}_{|k_1-l_j|}$, $\mathbb{Z}_{|k_2-l_j|}$ and $\mathbb{Z}_{|k_3-l_j|}$. After removing small equivariant neighborhoods of these orbits the action becomes free and the quotient is a smooth eight dimensional manifold $W_{k,l}$ with boundary $\partial(W_{k,l}) = E_{k,l} \cup L_1 \cup L_2 \cup L_3$ where the L_i are the following seven dimensional lens spaces.

$$\begin{aligned}
L_1 &:= L(k_1 - l_j; k_2 - l_j, k_3 - l_j, k_2 - l_{[j+1]_2}, k_3 - l_{[j+1]_2})\\
L_2 &:= L(k_2 - l_j; k_1 - l_j, k_3 - l_j, k_1 - l_{[j+1]_2}, k_3 - l_{[j+1]_2})\\
L_3 &:= L(k_3 - l_j; k_1 - l_j, k_2 - l_j, k_1 - l_{[j+1]_2}, k_2 - l_{[j+1]_2})
\end{aligned}$$

where we used the notation $[n]_p := m$ if $n = \lambda \cdot p + m$ for $m = 1, \ldots, p$, for the residue class $[n]$ modulo p.

Since the invariants are additive with respect to unions, we obtain $s_i(W_{k,l}) = s_i(E_{k,l}) + s_i(L_1) + s_i(L_2) + s_i(L_3) \in \mathbb{Q}/\mathbb{Z}$, $i = 1, 2$. Calculating $s_i(W_{k,l})$, yields the following expressions for the Kreck–Stolz invariants, which hold in the case condition (C) is satisfied for the jth column.

$$\begin{aligned}
s_1(E_{k,l}) = &\ \frac{4|r(k,l)(k_1 - l_j)(k_2 - l_j)(k_3 - l_j)| - q(k,l)^2}{2^7 \cdot 7 \cdot r(k,l)(k_1 - l_j)(k_2 - l_j)(k_3 - l_j)}\\
&- \sum_{i=1}^{3} s_1\left(k_{[i]_3} - l_j; k_{[i+1]_3} - l_j, k_{[i+2]_3} - l_j,\right.\\
&\qquad \left. k_{[i+1]_3} - l_{[j+1]_2}, k_{[i+2]_3} - l_{[j+1]_2}\right)\\
s_2(E_{k,l}) = &\ \frac{q(k,l) - 2}{2^4 \cdot 3 \cdot r(k,l)(k_1 - l_j)(k_2 - l_j)(k_3 - l_j)}\\
&- \sum_{i=1}^{3} s_2\left(k_{[i]_3} - l_j; k_{[i+1]_3} - l_j, k_{[i+2]_3} - l_j,\right.\\
&\qquad \left. k_{[i+1]_3} - l_{[j+1]_2}, k_{[i+2]_3} - l_{[j+1]_2}\right)
\end{aligned} \tag{2.1}$$

where $q(k,l) := (k_1 - l_j)^2 + (k_2 - l_j)^2 + (k_3 - l_j)^2 + (k_1 - l_{[j+1]_2})^2 + (k_2 - l_{[j+1]_2})^2 + (k_3 - l_{[j+1]_2})^2 - (l_j - l_{[j+1]_2})^2$, and $s_i(p; p_1, p_2, p_3, p_4) := s_i(L(p; p_1, p_2, p_3, p_4)) \in \mathbb{Q}/\mathbb{Z}$, $i = 1, 2$ are the Kreck–Stolz invariants of the lens space $L(p; p_1, p_2, p_3, p_4) = S^7/\mathbb{Z}_p$.

The freeness condition of the action (1.1) implies that $k_i - l_j \neq 0$ for $i, j = 1, 2, 3$, and hence the above expressions for s_1 and s_2 are well-defined.

By the Atiyah–Patodi–Singer index theorem the Kreck–Stolz invariants can also be expressed as linear combinations of eta-invariants. Calculating these eta-invariants for the lens spaces, see [15] for details, one obtains:

$$s_1(p; p_1, p_2, p_3, p_4) = \frac{1}{2^5 \cdot 7 \cdot p} \sum_{k=1}^{|p|-1} \prod_{j=1}^{4} \cot\left(\frac{k\pi p_j}{p}\right) + \frac{1}{2^4 \cdot p} \sum_{k=1}^{|p|-1} \prod_{j=1}^{4} \csc\left(\frac{k\pi p_j}{p}\right); \tag{2.2}$$

$$s_2(p; p_1, p_2, p_3, p_4) = \frac{1}{2^4 \cdot p} \sum_{k=1}^{|p|-1} \left(e^{\frac{2\pi \iota k}{|p|}} - 1\right) \prod_{j=1}^{4} \csc\left(\frac{k\pi p_j}{p}\right).$$

For $p = \pm 1$ these expressions are interpreted to be 0.

These formulas only hold in the case that condition (C) is satisfied for the jth column. We also list the case where the jth row consists of relatively prime entries, since this will be needed in our calculations.

$$s_1(E_{k,l}) = \frac{4\,|\,r(k,l)\,(k_j - l_1)\,(k_j - l_2)\,(k_j - l_3)\,| - q(k,l)^2}{2^7 \cdot 7 \cdot r(k,l)\,(k_j - l_1)\,(k_j - l_2)\,(k_j - l_3)} + \sum_{i=1}^{3} s_1\left(k_j - l_{[i]_3}; k_j - l_{[i+1]_3}, k_j - l_{[i+2]_3}, k_{[j+1]_2} - l_{[i+1]_3}, k_{[j+1]_2} - l_{[i+2]_3}\right)$$

$$s_2(E_{k,l}) = \frac{q(k,l) - 2}{2^4 \cdot 3 \cdot r(k,l)\,(k_j - l_1)\,(k_j - l_2)\,(k_j - l_3)} + \sum_{i=1}^{3} s_2\left(k_j - l_{[i]_3}; k_j - l_{[i+1]_3}, k_j - l_{[i+2]_3}, k_{[j+1]_2} - l_{[i+1]_3}, k_{[j+1]_2} - l_{[i+2]_3}\right)$$

where $q(k,l) := (k_j - l_1)^2 + (k_j - l_2)^2 + (k_j - l_3)^2 + (k_{[j+1]_2} - l_1)^2 + (k_{[j+1]_2} - l_2)^2 + (k_{[j+1]_2} - l_3)^2 - (k_j - k_{[j+1]_2})^2$.

Using these invariants we now state the classification theorems by Kruggel for the Eschenburg spaces.

Theorem 2.3 (*Kruggel*) [14,15] *Assume the Eschenburg spaces $E_{k,l}$ and $E_{k',l'}$ both satisfy condition* (*C*)*. Then*

(I) *$E_{k,l}$ and $E_{k',l'}$ are* (*orientation preserving*) *homeomorphic if and only if*

$$\begin{aligned} &(a) \quad |r(k,l)| = |r(k',l')| \in \mathbb{Z} \\ &(b) \quad \mathrm{Lk}(k,l) \equiv \mathrm{Lk}(k',l') \in \mathbb{Q}/\mathbb{Z} \\ &(c) \quad p_1(k,l) \equiv p_1(k',l') \in \mathbb{Z}_{|r(k,l)|} \\ &(d) \quad s_2(E_{k,l}) \equiv s_2(E_{k',l'}) \in \mathbb{Q}/\mathbb{Z} \end{aligned}$$

(II) *$E_{k,l}$ and $E_{k',l'}$ are (orientation preserving) diffeomorphic if and only if in addition*

$$s_1(E_{k,l}) \equiv s_1(E_{k',l'}) \in \mathbb{Q}/\mathbb{Z}$$

(III) *$E_{k,l}$ and $E_{k',l'}$ are (orientation preserving) homotopy equivalent if and only if*

$$\begin{aligned} &(a) \quad |r(k,l)| = |r(k',l')| \in \mathbb{Z} \\ &(b) \quad \mathrm{Lk}(k,l) \equiv \mathrm{Lk}(k',l) \in \mathbb{Q}/\mathbb{Z} \\ &(c) \quad s_{22}(E_{k,l}) \equiv s_{22}(E_{k',l'}) \in \mathbb{Q}/\mathbb{Z} \end{aligned}$$

where $s_{22}(E_{k,l}) := 2\,|r(k,l)|\,s_2(E_{k,l})$.

For the corresponding theorem in the orientation reversing case the linking number and the Kreck–Stolz invariants change signs. Recall that in this theorem, $r(k,l) = \sigma_2(k) - \sigma_2(l)$, $p_1(E_{k,l}) = [2\,\sigma_1(k)^2 - 6\,\sigma_2(k)] \in \mathbb{Z}_{|r|}$, and the equality of the linking forms can replaced by the equality of the numbers $s(k,l) := \sigma_3(k) - \sigma_3(l) \in \mathbb{Z}_{|r(k,l)|}$.

Remark 2.4 In [12] Kreck–Stolz used another invariant s_3 in the homeomorphism classification, and showed that $r(k,l)$, s_2 and s_3 determine the homeomorphism type. Following [15], the formula for the invariant s_3 for the Eschenburg spaces, assuming that condition (C) holds for the jth column, is easily seen to be:

$$\begin{aligned} s_3(E_{k,l}) = \; & \frac{q(k,l) - 8}{2^2 \cdot 3 \cdot r(k,l)\,(k_1 - l_j)\,(k_2 - l_j)\,(k_3 - l_j)} \\ & - \sum_{i=1}^{3} s_3\left(k_{[i]_3} - l_j; k_{[i+1]_3} - l_j, k_{[i+2]_3} - l_j,\right. \\ & \qquad \left. k_{[i+1]_3} - l_{[j+1]_2}, k_{[i+2]_3} - l_{[j+1]_2}\right) \end{aligned}$$

where

$$s_3(p; p_1, p_2, p_3, p_4) = \frac{1}{2^4 \cdot p} \sum_{k=1}^{|p|-1} \left(e^{\frac{4\pi\imath k}{|p|}} - 1\right) \prod_{j=1}^{4} \csc\left(\frac{k\pi p_j}{p}\right)$$

3 Number theory

The difficulty in explicitly calculating the invariants comes from the complicated expressions (2.2) for the Kreck–Stolz invariants of the lens spaces. These expressions are rational since they lie in $\mathbb{Q}/\mathbb{Z}$. However, in order to use a computer program to calculate the invariants it is necessary to control the denominators. The bounds on the denominators in the theorem below are similar to those obtained by Zagier in [19] for the higher order Dedekind sum T. However, for the other sums the results we need are not contained in [19], so we give a proof.

Theorem 3.1 *For all integers $p, p_1, \ldots, p_4$ such that $|p| > 1$ and p is relatively prime to each p_i, the numbers*

$$T = \sum_{k=1}^{|p|-1} \prod_{j=1}^{4} \cot\left(\frac{k\pi p_j}{p}\right), \quad S = \sum_{k=1}^{|p|-1} \prod_{j=1}^{4} \csc\left(\frac{k\pi p_j}{p}\right)$$
$$R = \sum_{k=1}^{|p|-1} \cos\left(\frac{2\pi k}{|p|}\right) \prod_{j=1}^{4} \csc\left(\frac{k\pi p_j}{p}\right) \quad \text{and}$$
$$U = \sum_{k=1}^{|p|-1} \cos\left(\frac{4\pi k}{|p|}\right) \prod_{j=1}^{4} \csc\left(\frac{k\pi p_j}{p}\right)$$

are rational with denominators which divide 45. *If $\sum_{i=1}^{4} p_i$ is even,*

$$\sum_{k=1}^{|p|-1} \left(e^{\frac{2\pi \imath k}{|p|}} - 1\right) \prod_{j=1}^{4} \csc\left(\frac{k\pi p_j}{p}\right) = R - S \quad \text{and}$$
$$\sum_{k=1}^{|p|-1} \left(e^{\frac{4\pi \imath k}{|p|}} - 1\right) \prod_{j=1}^{4} \csc\left(\frac{k\pi p_j}{p}\right) = U - S.$$

To begin the proof, note that the last two equalities are clear from grouping terms for k and $|p| - k$. For the rest of the proof, we can assume that $p > 0$ after replacing p by $-p$ if necessary.

Define

$$F_1(x) = \prod_{j=1}^{4} \cot(xp_j\pi), \quad F_2(x) = \prod_{j=1}^{4} \csc(xp_j\pi)$$
$$F_3(x) = \cos(2x\pi)F_2(x), \quad F_4(x) = \cos(4x\pi)F_2(x). \tag{3.2}$$

If F is one of the F_i then $F(x) = F(2 - x)$. It follows that

$$\sum_{k=1,k\neq p}^{2p-1} F(k/p) = 2T,\ 2S,\ 2R,\ 2U \quad \text{if } F = F_1,\ F_2,\ F_3,\ F_4, \text{respectively.} \tag{3.3}$$

For each divisor p' of $2p$ define

$$\mathcal{C}_{p'} = \{k : 1 \le k < 2p \quad \text{and} \quad (k, 2p) = p'\}$$
$$\mathcal{D}_d = \{k' : 1 \le k' < d \quad \text{and} \quad (k', d) = 1\} \tag{3.4}$$

The map $k' \to k' \cdot p'$ defines a bijection between $\mathcal{D}_d$ and $\mathcal{C}_{p'}$ for $d = \frac{2p}{p'}$. Therefore for $F \in \{F_1, F_2, F_3, F_4\}$,

$$\sum_{k=1,k\neq p}^{2p-1} F(k/p) = \sum_{\substack{p'\neq p\\ p'|2p}} \sum_{k\in C_{p'}} F(k/p) \\ = \sum_{2\neq d|2p} \sum_{k'\in D_d} F(2k'/d) \tag{3.5}$$

In deducing the last line, we set $d = \frac{2p}{p'}$ and use the fact that $\mathcal{D}_1$ is empty.

Let $\zeta_d = \exp\left(\frac{2\pi i}{d}\right)$, so that ζ_d is a primitive d^{th} root of unity. For each divisor d of $2p$ with $d > 2$, denoted by $2 < d|2p$, there is a bijection $\mathcal{D}_d \to \mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q}) = G_d$ which sends $k' \in \mathcal{D}_d$ to the unique automorphism $\sigma_{k'}$ of $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}$ which takes ζ_d to $\zeta_d^{k'}$. From (3.2) we have

$$F_1(2k'/d) = \prod_{j=1}^{4} \frac{\zeta_d^{k'p_j} + \zeta_d^{-k'p_j}}{\zeta_d^{k'p_j} - \zeta_d^{-k'p_j}}; \quad F_2(2k'/d) = 16\prod_{j=1}^{4} \frac{1}{\zeta_d^{k'p_j} - \zeta_d^{-k'p_j}}$$
$$F_3(2k'/d) = \left(\frac{\zeta_d^{2k'} + \zeta_d^{-2k'}}{2}\right)F_2(2k'/d); \quad F_4(2k'/d) = \left(\frac{\zeta_d^{4k'} + \zeta_d^{-4k'}}{2}\right)F_2(2k'/d). \tag{3.6}$$

Therefore for $F \in \{F_1, F_2, F_3, F_4\}$ and $k' \in \mathcal{D}_d$ we have $F(2k'/d) \in \mathbb{Q}(\zeta_d)$ and $\sigma_{k'}(F(2/d)) = F(2k'/d)$. Hence

$$\sum_{k=1,k\neq p}^{2p-1} F(k/p) = \sum_{2<d|2p} \sum_{k'\in\mathcal{D}_d} F(2k'/d) = \sum_{2<d|2p} \mathrm{Tr_d}(F(2/d)) \tag{3.7}$$

where $d = \frac{2p}{p'}$ and $\mathrm{Tr_d}\colon \mathbb{Q}(\zeta_d) \to \mathbb{Q}$ is the trace function defined by $\mathrm{Tr_d}(\tau) = \sum_{\sigma\in G_d} \sigma(\tau)$.

Define

$$\alpha_d = \prod_{j=1}^{4}\left(\zeta_d^{p_j} - \zeta_d^{-p_j}\right), \quad \beta_d = \prod_{j=1}^{4}\left(\zeta_d^{p_j} + \zeta_d^{-p_j}\right), \\ \gamma_d = \zeta_d^2 + \zeta_d^{-2}, \quad \tau_d = \zeta_d^4 + \zeta_d^{-4}. \tag{3.8}$$

Putting together (3.3), (3.7) and (3.6) shows

$$\begin{aligned} 2T &= \sum_{2<d|2p} \mathrm{Tr_d}\left(\frac{\beta_d}{\alpha_d}\right), \quad 2S = 16 \sum_{2<d|2p} \mathrm{Tr_d}\left(\frac{1}{\alpha_d}\right), \\ 2R &= 8 \sum_{2<d|2p} \mathrm{Tr_d}\left(\frac{\gamma_d}{\alpha_d}\right), \quad 2U = 8 \sum_{2<d|2p} \mathrm{Tr_d}\left(\frac{\tau_d}{\alpha_d}\right). \end{aligned} \tag{3.9}$$

Lemma 3.10 *Fix $2 < d|2p$ and let $d' = d/(d,2) > 1$. Then each of α_d, β_d, γ_d and τ_d are integers in the real subfield $\mathbb{Q}(\zeta_{d'})^+ = \mathbb{Q}(\zeta_{d'} + \zeta_{d'}^{-1})$ of $\mathbb{Q}(\zeta_{d'})$. Moreover,*

1. *If d' is not a prime power, then α_d is a unit.*
2. *Suppose $d' = l^r$ for some prime l and some $r > 0$. Let $\mathcal{P}$ be the unique prime ideal over l in $\mathbb{Q}(\zeta_{d'})^+$. If $d' \neq 2$ then α_d generates the ideal $\mathcal{P}^2$, while if $d' = 2$ then α_d generates $\mathcal{P}^4$. The relative degree $[\mathbb{Q}(\zeta_d) : \mathbb{Q}(\zeta_{d'})^+]$ is 2 if $l > 2$ or $d' = 2$, and $[\mathbb{Q}(\zeta_d) : \mathbb{Q}(\zeta_{d'})^+] = 4$ if $l = 2$ and $d' > 2$.*

Proof Let us first show α_d, β_d, γ_d and τ_d are integers in $\mathbb{Q}(\zeta_{d'})^+$. Since ζ_d is integral, so are α_d, β_d, γ_d and τ_d. Complex conjugation acts on $\mathbb{Q}(\zeta_d)$ by sending ζ_d to ζ_d^{-1}. Hence (3.8) shows that each of α_d, β_d, γ_d and τ_d are in $\mathbb{Q}(\zeta_d)^+$. If 4 does not divide d, then $\mathbb{Q}(\zeta_d)^+ = \mathbb{Q}(\zeta_{d'})^+$. Suppose now that $4|d$, so $d' = d/2$ is even. The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_{d'}))$ is generated by the automorphism σ which sends ζ_d to $\zeta_d^{1+d'} = -\zeta_d$. In this case, each of the p_j are odd since they are relatively prime to p, $2|(d/2)$ and $(d/2)|p$. It now follows from (3.8) that each of α_d, β_d, γ_d and τ_d are fixed by σ. Thus these numbers are in the real subfield $\mathbb{Q}(\zeta_{d'})^+ = \mathbb{Q}(\zeta_{d'}) \cap \mathbb{Q}(\zeta_d)^+$ in all cases.

We now prove the remaining assertions about α_d. In $\mathbb{Q}(\zeta_d)$ we have

$$\zeta_d^{p_j} - \zeta_d^{-p_j} = \zeta_d^{-p_j}\left(\zeta_d^{2p_j} - 1\right) = \zeta_d^{-p_j}\left(\zeta_{d'}^{p'_j} - 1\right) \tag{3.11}$$

where $d' = d$ and $p'_j = 2p_j$ if d is odd, and $d' = d/2$ and $p'_j = p_j$ if d is even. Recall that p_j is prime to p, and $d|2p$. Hence $p'_j = 2p_j$ is prime to $d' = d$ if d is odd, while $p'_j = p_j$ is prime to $d' = (d/2)|p$ if d is even. Thus $\zeta_{d'}^{p'_j}$ is a primitive d'^{th} root of unity in all cases. In [18, Proposition 2.8] it is shown that $1 - \zeta_{d'}^{p'_j}$ is a unit if d' has at least two prime factors, so α_d is a unit in this case. Note that $d' > 1$ since $d > 2$.

The remaining possibility is that $d' = l^r$ for some prime l and some $r > 0$. Then $1 - \zeta_{d'}^{p'_j}$ generates the unique prime ideal $\mathcal{Q}$ over l in $\mathbb{Q}(\zeta_{d'}) = \mathbb{Q}(\zeta_{l^r})$ by [18, p. 9]. From (3.8) and (3.11) we see that $\alpha_d \in \mathbb{Q}(\zeta_{d'})^+$ is the product $\zeta\alpha'$ of a root of unity $\zeta \in \mathbb{Q}(\zeta_d)$ with a generator $\alpha' \in \mathbb{Q}(\zeta_{d'})$ for $\mathcal{Q}^4$. Hence α_d generates $\mathcal{Q}^4$ in $\mathbb{Q}(\zeta_{d'})$. The degree $e = [\mathbb{Q}(\zeta_{d'}) : \mathbb{Q}(\zeta_{d'})^+]$ is 2 unless $d' = 2$, in which case $e = 1$. Since $\alpha_d \in \mathbb{Q}(\zeta_{d'})^+$, and $\mathcal{P} \subset \mathbb{Q}(\zeta_{d'})^+$ generates the ideal $\mathcal{Q}^e$ in $\mathbb{Q}(\zeta_{d'})$, we conclude that α_d generates the ideal $\mathcal{P}^2$ in $\mathbb{Q}(\zeta_{d'})^+$ unless $d' = 2$, in which case α_d generates $\mathcal{P}^4 = \mathcal{Q}^4$ in $\mathbb{Q}(\zeta_{d'})^+ = \mathbb{Q}(\zeta_{d'}) = \mathbb{Q}$. Since $d' = l^r > 1$, the extension $\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_{d'})^+$ is totally ramified over l, and has degree 2 unless $d' = 2^r > 2$, in which case it has degree 4.

□

Lemma 3.12 *With the notations of Lemma 3.10, each of* $\mathrm{Tr}_d(\frac{\beta_d}{\alpha_d})$, $\mathrm{Tr}_\mathrm{d}(\frac{1}{\alpha_d}t)$, $\mathrm{Tr}_\mathrm{d}(\frac{\gamma_d}{\alpha_d})$ *and* $\mathrm{Tr}_\mathrm{d}(\frac{\tau_d}{\alpha_d})$ *lie in* $2\mathbb{Z}$ *unless* $d' \in \{2, 3, 5\}$. *Moreover, for the remaining cases we obtain*:

1. *Suppose* $d' = 2$. *Then* $\mathrm{Tr}_d(\frac{\beta_d}{\alpha_d}) = 0$, $\mathrm{Tr}_d(\frac{1}{\alpha_d})$ *generates the ideal* $\frac{1}{8}\mathbb{Z}$, *and each of* $\mathrm{Tr}_d(\frac{\gamma_d}{\alpha_d}) = \frac{-4}{\alpha_d}$ *and* $\mathrm{Tr}_d(\frac{\tau_d}{\alpha_d}) = \frac{4}{\alpha_d}$ *generate the ideal* $\frac{1}{4}\mathbb{Z}$.
2. *Suppose* $d' = 3$. *Then* $\mathrm{Tr}_d(\frac{\beta_d}{\alpha_d})$ *lies in* $\frac{2}{9}\mathbb{Z}$, *while* $\mathrm{Tr}_d(\frac{1}{\alpha_d})$ *and* $\mathrm{Tr}_d(\frac{\gamma_d}{\alpha_d}) = \mathrm{Tr}_d(\frac{\tau_d}{\alpha_d})$ *each generate the ideal* $\frac{2}{9}\mathbb{Z}$
3. *Suppose* $d' = 5$. *Then* $\mathrm{Tr}_d(\frac{1}{\alpha_d})$, $\mathrm{Tr}_d(\frac{1}{\alpha_d})$, $\mathrm{Tr}_d(\frac{\gamma_d}{\alpha_d})$ *and* $\mathrm{Tr}_d(\frac{\tau_d}{\alpha_d})$ *lie in* $\frac{2}{5}\mathbb{Z}$.

Proof Suppose first that $d' = 2$. Then $d = 4$, and each p_j must be odd. Now $\zeta_d^{p_j} + \zeta_d^{-p_j} = i^{p_j} + i^{-p_j} = 0$, so $\beta_d = 0$ and $\mathrm{Tr}_d(\frac{\beta_d}{\alpha_d}) = 0$. We have $[\mathbb{Q}(\zeta_d) : \mathbb{Q}(\zeta_{d'})^+] = 2$. The number α_d generates the ideal $2^4\mathbb{Z}$ in $\mathbb{Q}(\zeta_{d'})^+ = \mathbb{Q}$, $\gamma_d = \zeta_d^2 + \zeta_d^{-2} = -2$ and $\tau_d = \zeta_d^4 + \zeta_d^{-4} = 2$. Hence $\mathrm{Tr}_d(\frac{1}{\alpha_d}) = \frac{2}{\alpha_d}$ generates the ideal $\frac{1}{8}\mathbb{Z}$, while $\mathrm{Tr}_d(\frac{\gamma_d}{\alpha_d}) = \frac{-4}{\alpha_d}$ and $\mathrm{Tr}_d(\frac{\tau_d}{\alpha_d}) = \frac{4}{\alpha_d}$ each generate the ideal $\frac{1}{4}\mathbb{Z}$.

Now suppose $d' = 3$. Then $d = 3$ or $d = 6$, so $[\mathbb{Q}(\zeta_d) : \mathbb{Q}(\zeta_{d'})^+] = 2$. The number α_d generates the ideal $3^2\mathbb{Z}$ in $\mathbb{Q}(\zeta_{d'})^+ = \mathbb{Q}$, and $\gamma_d = \zeta_d^2 + \zeta_d^{-2} = -1 = \zeta_d^4 + \zeta_d^{-4} = \tau_d$. So $\mathrm{Tr}_d(\frac{\beta_d}{\alpha_d}) = \frac{2\beta_d}{\alpha_d}$ lies in the ideal $\frac{2}{9}\mathbb{Z}$, while $\mathrm{Tr}_d(\frac{1}{\alpha_d}) = \frac{2}{\alpha_d}$ and $\mathrm{Tr}_d(\frac{\gamma_d}{\alpha_d}) = \frac{-2}{\alpha_d} = \mathrm{Tr}_d(\frac{\tau_d}{\alpha_d})$ each generate the ideal $\frac{2}{9}\mathbb{Z}$.

Next we consider $d' = 5$. Then $\mathbb{Q}(\zeta_{d'})^+ = \mathbb{Q}(\sqrt{5}) = L$ is quadratic over $\mathbb{Q}$, and α_d generates the square of the unique prime $\mathcal{P}$ over $l = 5$ in this extension. Since $\mathcal{P}^2$ is generated by 5, we conclude that for $\xi \in \{\frac{\beta_d}{\alpha_d}, \frac{1}{\alpha_d}, \frac{\gamma_d}{\alpha_d}, \frac{\tau_d}{\alpha_d}\}$, the number 5ξ is integral in L. This implies $\mathrm{Tr}_d(\xi) = \mathrm{Tr}_{L/\mathbb{Q}}(2\xi) = \frac{2}{5}\mathrm{Tr}_{L/\mathbb{Q}}(5\xi) \in \frac{2}{5}\mathbb{Z}$, as claimed.

In the remaining computations we suppose $\xi \in \{\beta_d, 1, \gamma_d, \tau_d\}$, $d' > 3$ and $d' \neq 5$. If d' is not a prime power then α_d is a unit, so $\frac{\xi}{\alpha_d}$ is integral in $\mathbb{Q}(\zeta_{d'})^+$. If $d' = l^r$ for some prime l and some $r > 0$, then either $l > 5$ or $r \geq 2$, and α_d generates the square of the unique prime $\mathcal{P}$ over l in $\mathbb{Q}(\zeta_{d'})^+$. In this case, the degree of $\mathbb{Q}(\zeta_{d'})^+ = \mathbb{Q}(\zeta_{l^r})^+$ over $\mathbb{Q}$ is $\phi(l^r)/2 = (l-1)l^{r-1}/2 \geq 3$, so $\frac{\xi}{\alpha_d} \in \mathcal{P}^{-2}$ and $\mathcal{P}^{-2}$ is contained in the inverse different of $\mathbb{Q}(\zeta_{d'})^+ = \mathbb{Q}(\zeta_{l^r})^+$ over $\mathbb{Q}$. We conclude that in all cases, $\mathrm{Tr}_{d'+}(\frac{\xi}{\alpha_d}) \in \mathbb{Z}$, where $\mathrm{Tr}_{d'+}$ is the trace from $\mathbb{Q}(\zeta_{d'})^+$ to $\mathbb{Q}$. Hence

$$\mathrm{Tr}_d\left(\frac{\xi}{\alpha_d}\right) = \mathrm{Tr}_{d'+}\left(\frac{[\mathbb{Q}(\zeta_d) : \mathbb{Q}(\zeta_{d'})^+]\xi}{\alpha_d}\right) = \left[\mathbb{Q}(\zeta_d) : \mathbb{Q}(\zeta_{d'})^+\right] \cdot \mathrm{Tr}_{d'+}\left(\frac{\xi}{\alpha_d}\right)$$

is in $2\mathbb{Z}$ since $[\mathbb{Q}(\zeta_d) : \mathbb{Q}(\zeta_{d'})^+]$ is even. □

Combining these results yields the following proposition, which proves Theorem 3.1.

Proposition 3.13 *The rational numbers* T, R, S, *and* U *from Theorem 3.1 satisfy the following divisibility properties:*

1. *If* $(p, 3) = (p, 5) = 1$, *then* T, R, S, U *are integers.*
2. *If* $(p, 3) = 3$ *and* $(p, 5) = 1$, *then the denominators are divisors of* 9.

3. *If* $(p, 3) = 1$ *and* $(p, 5) = 5$*, then the denominators are divisors of* 5.
4. *If* $(p, 3) = 3$ *and* $(p, 5) = 5$*, then the denominators are divisors of* 45.

In terms of T, R, S, U the Kreck–Stolz invariants of the lens spaces are given by:

$$s_1 = \frac{1}{2^5 \cdot 7 \cdot p}(T + 14S)\,, \; s_2 = \frac{1}{2^4 \cdot p}(R - S)\,, \; s_3 = \frac{1}{2^4 \cdot p}(U - S) \quad (3.14)$$

In order to determine the values of these Kreck–Stolz invariants on a computer, we multiply T, R, S, U by 45, find an integer approximation, and use (3.14).

4 Examples

Using a program written in Maple and C code we generate the following lists of examples. The program is available at http://www.math.upenn.edu/wziller/research, and can be described briefly as follows. For each given odd order $r = |r(k, l)| = |\sigma_2(k) - \sigma_2(l)| < 50,000$ we produce a list of all positively curved Eschenburg spaces with that given order of the fourth cohomology group. In total, there are 26, 330, 623 positively curved Eschenburg spaces with $r < 50,000$. In the next step the program computes the basic polynomial invariants $s(k, l)$ and $p_1(k, l)$ and produces a list of pairs whose basic invariants coincide. The program also checks condition (C) and finds that it is always satisfied for such pairs. Generating the list and comparing the basic invariants are very time and memory intensive calculations which forced us to write the program in C code. Surprisingly, there are only 437 pairs of spaces whose basic invariants coincide.

For this significantly smaller list of spaces the Kreck–Stolz invariants are computed and compared, which can be done in Maple.

We also indicate the cohomogeneity of the examples in the last column. Here 2+ denotes the cohomogeneity two Eschenburg spaces with $k_1 = k_2$, containing the 3-Sasakian spaces, and 2− the case of $l_1 = l_2$.

We list the invariant $p_1 \in \mathbb{Z}_r$ as lying in the interval $[0, r-1]$, the (orientation sensitive) invariants $s(k, l) \in \mathbb{Z}_r$, which describes the linking form, as lying in $(-\frac{r-1}{2}, \frac{r-1}{2}]$ and $s_1, s_2, s_{22} \in \mathbb{Q}/\mathbb{Z}$ as lying in $(-\frac{1}{2}, \frac{1}{2}]$. The advantage of choosing these intervals is that one sees immediately when the invariants just differ by a sign and hence the corresponding spaces are orientation reversing homeomorphic or diffeomorphic.

We first produce a list of homotopy equivalent positively curved Eschenburg spaces in Table 1 for $r \le 200$. Such examples occur very frequently, e.g. there are 192 such pairs for $r < 1,000$. See [16] for the first examples of this type in the literature.

In order to find pairs of homeomorphic or diffeomorphic Eschenburg spaces we increased r to 50, 000. There are 69 homeomorphic pairs, the first 5 of which are listed in Table 2, and only four diffeomorphic pairs, which we list in Table 3. It is interesting to note that for $r = 26,973$ there are two Eschenburg metrics on the same manifold, one cohomogeneity two and the other cohomogeneity four. It follows from [11] that the corresponding Eschenburg metrics cannot be isometric to each other since they have different full isometry group.

Table 1 Homotopy equivalent Eschenburg spaces for $r < 200$

r	$[k_1, k_2, k_3 \mid l_1, l_2, l_3]$	s	s_{22}	p_1	Cohom
43	[21, 21, −2 | 20, 20, 0]	21	−1/6	26	1
43	[8, 7, −5 | 6, 4, 0]	21	−1/6	13	4
101	[50, 50, −2 | 49, 49, 0]	50	−1/6	55	1
101	[12, 10, −8 | 9, 5, 0]	50	−1/6	21	4
137	[68, 68, −2 | 67, 67, 0]	68	−1/6	73	1
137	[19, 17, −7 | 16, 13, 0]	68	−1/6	23	4
181	[16, 16, −10 | 13, 9, 0]	−26	−1/6	85	2+
181	[30, 26, −6 | 25, 25, 0]	26	1/6	164	2−
181	[45, 43, −4 | 42, 42, 0]	43	0	89	2−
181	[15, 14, −11 | 12, 6, 0]	43	0	35	4

Table 2 Homeomorphic Eschenburg spaces for $r < 12{,}000$

r	$[k_1, k_2, k_3 \mid l_1, l_2, l_3]$	s	$[p_1, s_2]$	s_1	Cohom
4,001	[79, 49, −50 | 46, 32, 0]	−1502	[3336, −1043/8002]	49741/112028	4
4001	[75, 54, −51 | 46, 32, 0]	1502	[3336, 1043/8002]	1877/8002	4
8,099	[71, 59, −94 | 34, 2, 0]	3085	[2184, −6975/32396]	−1055/9968	4
8,099	[92, 47, −85 | 38, 16, 0]	−3085	[2184, 6975/32396]	−4285/9968	4
8,671	[83, 43, −96 | 24, 6, 0]	4216	[936, −11343/34684]	−941/10672	4
8,671	[97, 33, −88 | 24, 18, 0]	−4216	[936, 11343/34684]	−1417/74704	4
9,889	[104, 96, −86 | 81, 33, 0]	1719	[65, 9505/39556]	2961/79112	4
9,889	[109, 101, −81 | 81, 48, 0]	−1719	[65, −9505/39556]	275943/553784	4
11,011	[144, 136, −76 | 135, 69, 0]	−1899	[5320, −6767/22022]	31695/176176	4
11,011	[152, 144, −68 | 129, 99, 0]	−1899	[5320, −6767/22022]	12819/176176	4

Table 3 Diffeomorphic Eschenburg spaces for $r \leq 50{,}000$

r	$[k_1, k_2, k_3 \mid l_1, l_2, l_3]$	s	$[p_1, s_2]$	s_1	Cohom
13,361	[145, 121, −89 | 113, 64, 0]	1732	[5905, 6839/53444]	−272959/748216	4
13,361	[151, 127, −83 | 104, 91, 0]	−1732	[5905, −6839/53444]	272959/748216	4
26,973	[154, 154, −158 | 135, 15, 0]	2119	[5877, 123965/323676]	−6131/18648	2+
26,973	[389, 383, −67 | 357, 348, 0]	−2119	[5877, −123965/323676]	6131/18648	4
35,749	[185, 115, −186 | 102, 12, 0]	10989	[18648, 8920/35749]	−9018/35749	4
35,749	[230, 111, −155 | 108, 78, 0]	10989	[18648, 8920/35749]	−9018/35749	4
42,319	[205, 141, −193 | 114, 39, 0]	7443	[20142, 4123/84638]	−73317/677104	4
42,319	[191, 157, −195 | 114, 39, 0]	−7443	[20142, −4123/84638]	73317/677104	4

Next, we specialize to the subfamily of cohomogeneity two Eschenburg spaces which are 3-Sasakian, $E_{a,b,c} = \mathrm{diag}(z^a, z^b, z^c)\backslash \mathrm{SU}(3)/\mathrm{diag}(z^{a+b+c}, 1, 1)$. Homotopy equivalent 3-Sasakian spaces again exist in abundance. We list the first 5 examples with $r < 2,000$ in Table 4.

Table 4 Homotopy equivalent 3-Sasakian spaces $r < 2{,}000$

r	$[a, b, c \mid a+b+c, 0, 0]$	s	s_{22}	p_1
1,267	[316, 3, 1 | 320, 0, 0]	−319	1/3	813
1,267	[25, 19, 18 | 62, 0, 0]	−319	1/3	86
1,277	[181, 5, 2 | 188, 0, 0]	533	1/6	453
1,277	[44, 19, 7 | 70, 0, 0]	−533	−1/6	861
1,557	[778, 1, 1 | 780, 0, 0]	778	1/6	783
1,557	[139, 7, 4 | 150, 0, 0]	778	1/6	1404
1,595	[398, 3, 1 | 402, 0, 0]	−401	0	1018
1,595	[36, 23, 13 | 72, 0, 0]	−401	0	798
1,619	[105, 11, 4 | 120, 0, 0]	−237	0	1277
1,619	[132, 7, 5 | 144, 0, 0]	−237	0	997

Table 5 Homeomorphic 3-Sasakian spaces for $r < 500{,}000$

r	$[a, b, c \mid a+b+c, 0, 0]$	s	$[p_1,\ s_2]$	s_1
28,379	[171, 164, 1 | 336, 0, 0]	−335	[27139, −2393/56758]	−82869/3178448
28,379	[223, 60, 53 | 336, 0, 0]	−335	[27139, −2393/56758]	−1104513/3178448
1,29,503	[362, 291, 37 | 690, 0, 0]	12564	[45679, −80901/259006]	69409/14504336
129,503	[423, 169, 98 | 690, 0, 0]	12564	[45679, −80901/259006]	5767541/14504336
273,581	[717, 362, 13 | 1,092, 0, 0]	91230	[196280, 370663/1094324]	−393315/1094324
273,581	[761, 241, 90 | 1,092, 0, 0]	91230	[196280, 370663/1094324]	310179/1094324
382,025	[891, 368, 43 | 1,302, 0, 0]	−35741	[334208, −294993/1528100]	−74669/436600
382,025	[928, 191, 183 | 1,302, 0, 0]	−35741	[334208, −294993/1528100]	1442017/3056200
442,179	[1265, 347, 2 | 1,614, 0, 0]	−6448	[346023, 115166/1326537]	−173889/611408
442,179	[1274, 311, 29 | 1,614, 0, 0]	−6448	[346023, 115166/1326537]	−21037/611408

Table 6 Diffeomorphic 3-Sasakian spaces for $r < 10^7$

r	$[a, b, c \mid 0, 0, a+b+c]$	s	$[p_1,\ s_2]$	s_1
5,143,925	[2279, 1603, 384 | 4266, 0, 0]	−1448517	[390037, 36777/4115140]	−37291099/144029900
5,143,925	[2528, 939, 799 | 4266, 0, 0]	−1448517	[390037, 36777/4115140]	−37291099/144029900

Our main goal for this subfamily was to produce two 3-Sasakian spaces which are diffeomorphic to each other, see Theorem A. To do this we increased r to 10^7. There are 3, 167, 786, 351 3-Sasakian spaces with $r < 10^7$ and among those only 290 pairs with the same basic invariants $s(k, l)$ and $p_1(k, l)$. For these pairs we then computed the Kreck–Stolz invariants. There are 139 pairs of homeomorphic 3-Sasakian spaces, the first five given in Table 5, but only one diffeomorphic pair, see Table 6. A peculiar fact for the homeomorphic pairs is that the sum $a + b + c$ and the orientation is always the same. But we were not able to see this directly. Notice also that this is not true for homotopy equivalent pairs of 3-Sasakian spaces, or for homeomorphic pairs of Eschenburg spaces.

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