Discriminants and Arakelov Euler Characteristics

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1 Introduction

The study of discriminants has been a central part of algebraic number theory (c.f. [30]), and has recently led to striking results in arithmetic geometry (e.g. [6], [33], [2]). In this article we summarize two different generalizations ([16], [11]) of discriminants to arithmetic schemes having a tame action by a finite group. We also discuss the results proved in [11, 15, 16] concerning the connection of these discriminants to ϵ_0 and ϵ-factors in the functional equations of L-functions. These results relate invariants defined by coherent cohomology (discriminants) to ones defined by means of étale cohomology (conductors and ϵ-factors.) One consequence is a proof of a conjecture of Bloch concerning the conductor of an arithmetic scheme [15] when this scheme satisfies certain hypotheses (c.f. Theorem 2.6.4). In the last section of this paper we present an example involving integral models of elliptic curves.

The discriminant d_K of a number field K can be defined in (at least) three different ways. The definition closest to Arakelov theory arises from the fact that $\sqrt{|d_K|}$ is the covolume of the ring of integers O_K of K in $\mathbf{R} \otimes_{\mathbf{Q}} K$ with respect to a natural Haar measure on $\mathbf{R} \otimes_{\mathbf{Q}} K$ (c.f. [8, §4]). This Haar measure is the one which arises from the usual metrics at infinity one associates to $\mathrm{Spec}(K)$ as an arithmetic variety. A second definition of d_K is that it is the discriminant of the bilinear form defined by the trace function $\mathrm{Tr}_{K/\mathbf{Q}} : K \to \mathbf{Q}$. The natural context in which to view this definition is in terms of the coherent duality theorem, since $\mathrm{Tr}_{K/\mathbf{Q}}$ is the trace map which the duality theorem associates to the finite morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathbf{Q})$. A third definition of the ideal $d_K\mathbf{Z}$ is that this is the norm of the annihilator

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of the sheaf $\Omega^1_{O_K/\mathbf{Z}}$ of relative differentials on $\mathrm{Spec}(O_K)$. Each of these definitions is linked to the behavior of the zeta function $\zeta_K(s)$ of K via the fact that $|d_K|^{s/2}$ appears as a factor in the functional equation of $\zeta_K(s)$ (c.f. [29, p. 254]). This fact reflects that $|d_K|$ is also equal to the Artin conductor of the (permutation) representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the complex vector space $\oplus_{\sigma:K\to\overline{\mathbf{Q}}} \mathbf{C}\cdot e_\sigma$

Let $\mathcal{X}$ be a regular scheme which is flat and projective over $\mathbf{Z}$ and equidimensional of dimension d. In this case, a connection between the conductor in the conjectural functional equation of the Hasse-Weil L-function of $\mathcal{X}$ and an invariant involving the differentials of $\mathcal{X}$ has been conjectured by S. Bloch ([5]). In the case of curves (when $d = 1$), Bloch gave in [6] an unconditional proof of his conjecture in [5].

In this paper we will consider Artin-Hasse-Weil L-functions by letting a finite group G act (tamely) on $\mathcal{X}$, in the sense that the order of the inertia group $I_x \subset G$ of each point $x \in \mathcal{X}$ is prime to the residue characteristic of x. We will explain how can one relate the conductors and epsilon factors of the Artin-Hasse-Weil L-functions associated to $\mathcal{X}$ and representations of G (which are defined via étale cohomology) to two kinds of discriminants associated to the G-action on $\mathcal{X}$. These discriminants are defined via Arakelov theory and coherent duality, respectively.

To apply Arakelov theory, we choose in §2 a G-invariant Kähler metric h on the tangent bundle of the associated complex manifold $\mathcal{X}(\mathbf{C})$. Let $\mathbf{Z}G$ be the integral group ring of G. We sketch in §2 the construction given in [16] of an Arakelov-Euler characteristic associated to a hermitian G-bundle $(\mathcal{F}, j)$ on $\mathcal{X}$. This construction proceeds by endowing the equivariant determinant of cohomology of $\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F})$ with equivariant Quillen metrics $j_{Q,\phi}$ for each irreducible character ϕ of G. The resulting Euler characteristic lies in an arithmetic classgroup $A(\mathbf{Z}G)$. This classgroup is a G-equivariant version of the Arakelov class group of metrized vector bundles on $\mathrm{Spec}(\mathbf{Z})$, which it is isomorphic to when G is the trivial group.

To generalize the connection between the discriminant of O_K and the functional equation $\zeta_K(s)$, we consider variants of the de Rham complex on $\mathcal{X}$. One complication is that in general, the sheaf $\Omega^i_{\mathcal{X}/\mathbf{Z}}$ of degree i relative differentials will not be locally free on $\mathcal{X}$. We thus start by considering instead of $\Omega^1_{\mathcal{X}/\mathbf{Z}}$ the sheaf $\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)$ degree one relative logarithmic differentials with respect to the union of the reductions of the fibers of $\mathcal{X}$ over a large finite set S of primes of $\mathrm{Spec}(\mathbf{Z})$ (c.f. [27]) . Under certain hypotheses (Hypothesis 2.4.1), $\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)$ is locally free of rank d as an $\mathcal{O}_{\mathcal{X}}$-module.

Using exterior powers of $\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)$ and the metrics induced by a choice of Kähler metric on $\mathcal{X}$, we define in §2.4 a logarithmic de Rham Euler characteristic $\chi_{dRl}(\mathcal{X}, G, S)$ in $A(\mathbf{Z}G)$. This Euler characteristic can be viewed as an Arakelov-theoretic discriminant associated to the action of G on X.

To connect $\chi_{dRl}(\mathcal{X}, G, S)$ to L-functions, we make a further simplification by considering only symplectic characters of G. A complex representation of G is symplectic if it has a G-invariant non-degenerate alternating bilinear form. The group R^s_G of virtual symplectic characters of G is the subgroup of the character group R_G of G generated by the characters of symplectic representations. In §2.3 we define using symplectic characters a quotient $A^s(\mathbf{Z}G)$ of $A(\mathbf{Z}G)$ called the symplectic arithmetic class group. This quotient has the advantage that it contains a subgroup $\mathcal{R}^s(\mathbf{Z}G)$ of so-called rational classes, which is naturally identified with $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})}(R^s_G, \mathbf{Q}^\times)$.

The main result discussed in §2.5 is that the image $\chi^s_{dRl}(\mathcal{X}, G, S)$ of $\chi_{dRl}(\mathcal{X}, G, S)$ in $A^s(\mathbf{Z}G)$ is a rational class which determines and is determined by certain ϵ_0-constants associated to the L-functions of the Artin motives obtained from $\mathcal{X}$ and the symplectic representations of G.

We will not discuss here a second result, proved in [16], concerning the actual de Rham complex of $\mathcal{X}$, or rather a canonical complex of coherent G-sheaves on $\mathcal{X}$ which results from applying a construction of derived exterior powers due to Dold and Puppe to the relative differentials $\Omega^1_{\mathcal{X}/\mathbf{Z}}$. We refer the reader to [16] for the proof that the Arakelov Euler characteristic of this complex, together with a 'ramification class' associated to the bad fibers of $\mathcal{X}$, determines the symplectic ϵ-constants of $\mathcal{X}$. This result provides a metrized generalization of the main results in [9, 12, 14], which concern a generalization to schemes of Fröhlich's conjecture concerning rings of integers. For a survey of work on the Galois module structure of schemes, see [18].

A key step in establishing the results in [16] is to consider the case in which G is the trivial group. This case reduces to a conjecture of Bloch if $\mathcal{X}$ satisfies Hypothesis 2.4.1, which was mentioned above in connection with $\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)$. We discuss in §2.6 a proof given in [15] for such $\mathcal{X}$. An independent proof of Bloch's conjecture for these $\mathcal{X}$ has been given by K. Arai in his thesis. A proof which does not require the assumption that the multiplicities of the irreducible components of the fibers of $\mathcal{X}$ are prime to the residue characteristic has been given by Kato and Saito (to appear). For general $\mathcal{X}$, we discuss in §2.6 the proof given in [15] that Bloch's conjecture is equivalent to a statement about a metrized Arakelov Euler characteristic.

In §3 we turn to the other method of generalizing discriminants, via the coherent duality theorem. In [11], an Euler characteristic in Fröhlich's Hermitian class group $HCl(\mathbf{Z}G)$ is associated to a perfect complex $P^\bullet$ of $\mathbf{Z}G$-modules for which one has certain G-invariant $\mathbf{Q}$-valued pairings in cohomology. This leads to an Euler characteristic in $HCl(\mathbf{Z}G)$ associated to the logarithmic de Rham complex and the natural pairings on the de Rham cohomology of the general fiber of $\mathcal{X}$ which arise from duality. We state in §3.5 the results of [11] for $\mathcal{X}$ of dimension 2. In this case, the image of the above class in Fröhlich's adelic Hermitian class group both determines and is determined by the ϵ_0 -factors of representations of G. We will refer the reader to [11] for an analogous result, when $\mathcal{X}$ has dimension 2, concerning the Dold-Puppe variant of the de Rham complex mentioned in connection with §2.

In a final section we discuss an example of the Euler characteristics results sketched in §2 and §3, in which $\mathcal{X}/G$ is a regular model of an elliptic curve having reduced special fibers.

2 Equivariant Arakelov Euler Characteristics.

2.1 The equivariant determinant of cohomology.

Let $C^\bullet$ denote a perfect complex of $\mathbf{C}G$-modules. Thus the terms of $C^\bullet$ are finitely generated (necessarily projective), and all but a finite number of the terms are zero. Let $\widehat{G}$ be the set of complex irreducible characters of G. For each irreducible character $\phi \in \widehat{G}$ let W_ϕ denote the simple 2-sided $\mathbf{C}G$-ideal with character $\phi(1)\,\overline{\phi}$, where $\overline{\phi}$ is the contragredient character of ϕ. For a finitely generated $\mathbf{C}G$-module M we define $M_\phi = (M \otimes_{\mathbf{C}} W)^G$, where G acts diagonally and on the left of each term. Define the complex line

$$\det\left(H^*\left(C^\bullet\right)_\phi\right) = \otimes_i \wedge^{top}\left(H^i\left(C^\bullet\right)_\phi\right)^{(-1)^i} \tag{2.1.1}$$

where for a finite dimensional vector space V of dimension d, $\wedge^{top}(V)^{-1}$ is the dual of the complex line $\wedge^d(V)$. We recall from [28] the fundamental fact that there is a canonical isomorphism of complex lines

$$\xi_\phi : \det\left(C^\bullet_\phi\right) \to \det\left(H^*\left(C^\bullet\right)_\phi\right). \tag{2.1.2}$$

By a complex conjugation on a complex line L we will mean an isomorphism $\lambda : L \to L$ of additive groups such that $\lambda(\alpha l) = \overline{\alpha} l$ for $\alpha \in \mathbf{C}$ and $l \in L$.

Definition 2.1.1 *The equivariant determinant of cohomology of $C^\bullet$ is the family of complex lines $\left\{\det\left(H^*\left(C^\bullet\right)_\phi\right)\right\}_{\phi\in\widehat{G}}$. A metrized perfect $\mathbf{Z}G$-complex is a pair $(P^\bullet, h_\bullet)$ where $P^\bullet$ is a perfect $\mathbf{Z}G$-complex, and for each the $\phi \in \widehat{G}$, h_ϕ is a metric on the complex line $\det\left(H^*\left(P^\bullet \otimes_{\mathbf{Z}} \mathbf{C}\right)_\phi\right)$. We will say that h is invariant under complex conjugation if on each complex line $\det\left(H^*\left(P^\bullet \otimes_{\mathbf{Z}} \mathbf{C}\right)_\phi\right)$, one has specified a complex conjugation under which h_ϕ is invariant.*

2.2 Equivariant degree.

Consider a metrized perfect $\mathbf{Z}G$-complex $(P^\bullet, h_\bullet)$ as above. If $G = \{1\}$, the degree of $(P^\bullet, h)$ is defined to be the positive real number

$$\chi\left(P^\bullet, h\right) = h\left(\otimes_i\left(\wedge_j p_{ij}\right)^{(-1)^i}\right) \in \mathbf{R}_{>0} \tag{2.2.1}$$

for any choice of basis $\{p_{ij}\}$ for P_i as a $\mathbf{Z}$-module.

When G is non-trivial, we cannot in general find bases for the P_i as $\mathbf{Z}G$-modules, since the P_i are only locally free. We therefore take the following adelic approach.

Let $J_f(\overline{\mathbf{Q}})$ denote the finite ideles of the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$, and let $\Omega = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Define $\mathrm{Hom}_\Omega\left(R_G, J(\overline{\mathbf{Q}})^+\right)$ to be the subgroup

$$\mathrm{Hom}_\Omega\left(R_G, J_f(\overline{\mathbf{Q}})\right) \times \mathrm{Hom}\left(R_G, \mathbf{R}_{>0}\right)$$

of $\mathrm{Hom}\left(R_G, J_f(\overline{\mathbf{Q}}) \times \mathbf{R}^\times\right)$. The equivariant degree $\chi\left(P^\bullet, h_\bullet\right)$ takes values in the equivariant arithmetic class group $A\left(\mathbf{Z}G\right)$, which is defined in [16] as the following quotient group of $\mathrm{Hom}_\Omega\left(R_G, J(\overline{\mathbf{Q}})^+\right)$.

Let $\widehat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$ denote the ring of integral finite ideles of $\mathbf{Z}$. For $x \in \widehat{\mathbf{Z}}G^\times$, the element $\mathrm{Det}(x) \in \mathrm{Hom}_\Omega(R_G, J_f)$ is defined by the rule that for a representation T with character ψ

$$\mathrm{Det}(x)(\psi) = \det(T(x));$$

the group of all such homomorphisms is denoted

$$\mathrm{Det}(\widehat{\mathbf{Z}}G^\times) \subseteq \mathrm{Hom}_\Omega(R_G, J_f).$$

More generally, for $n > 1$ we can form the group $\mathrm{Det}\left(GL_n(\widehat{\mathbf{Z}}G)\right)$; as each group ring $\mathbf{Z}_pG$ is semi-local we have the equality $\mathrm{Det}\left(GL_n(\widehat{\mathbf{Z}}G)\right) = \mathrm{Det}(\widehat{\mathbf{Z}}G^\times)$ (see 1.2.6 in [35]).

Replacing the ring $\widehat{\mathbf{Z}}$ by $\mathbf{Q}$, in the same way we construct

$$\mathrm{Det}(\mathbf{Q}G^\times) \subseteq \mathrm{Hom}_\Omega(R_G, \overline{\mathbf{Q}}^\times).$$

The product of the natural maps $\overline{\mathbf{Q}}^\times \to J_f$ and $|-| : \overline{\mathbf{Q}^\times} \to \mathbf{R}_{>0}$ yields an injection

$$\Delta : \mathrm{Det}(\mathbf{Q}G^\times) \to \mathrm{Hom}_\Omega(R_G, J_f) \times \mathrm{Hom}(\mathrm{R_G}, \mathbf{R}_{>0}).$$

Definition 2.2.1 *The arithmetic classgroup $A(\mathbf{Z}G)$ is defined to be the quotient group*

$$A(\mathbf{Z}G) = \left(\frac{\mathrm{Hom}_\Omega(R_G, J_f) \times \mathrm{Hom}(R_G, \mathbf{R}_{>0})}{\left(\mathrm{Det}(\widehat{\mathbf{Z}}G^\times) \times 1\right)\ \mathrm{Im}(\Delta)} \right). \tag{2.2.2}$$

For each prime ideal $v \in \mathrm{Spec}\,(\mathbf{Z})$, we choose local bases $\left\{p_{ij}^{(v)}\right\}$ of $P^i \otimes_{\mathbf{Z}} \mathbf{Z}_v$. If d_i is the $\mathbf{Z}G$ rank P^i, then for each prime number l we can find $\alpha_i^{(l)} \in GL_{d_i}(\mathbf{Q}_lG)$ such that

$$p_{ij}^{(0)} = \alpha_i^{(l)} p_{ij}^{(l)} \tag{2.2.3}$$

Let ν be the Hermitian form on $\mathbf{C}G$ defined by

$$\nu\left(\sum x_g g, \sum y_h h\right) = \#G \cdot \sum x_g \overline{y}_g$$

For a given irreducible character $\phi \in \widehat{G}$, choose an orthonormal basis $\{w_{\phi,k}\}$ of the ideal W_ϕ of $\mathbf{C}G$ defined in §2.1 with respect to ν.[1] For each i we shall write $b_{i,\phi}$ for the wedge product

$$\wedge_{j,k}\left(\sum_g g p_0^{ij} \otimes g w_{\phi,k}\right) \in \det\left(P^i \otimes W\right)^G.$$

[1]The form ν arises from the following fact proved in [16, Lemma 2.3]. Suppose M is a finitely generated $\mathbf{C}G$-module and $||\ ||$ is a G-invariant metric on $\det(M)$. Give $M_\phi = (M \otimes_{\mathbf{C}} W)^G$ the metric induced by the tensor product metric on $M \otimes_{\mathbf{C}} W$ associated to $||\ ||$ and ν. Then the map $M_\phi \to \overline{W}M$ defined by $\sum v_i \otimes m_i \to \sum_i \overline{v}_i m_i$ is an isometry when $\overline{W}M \subset M$ is given the metric induced from $||\ ||$.

Recall from (2.1.2) that for each $\phi \in \widehat{G}$ we have an isomorphism

$$\xi_\phi : \det\left(P_\phi^\bullet\right) \to \det\left(H^*\left(P^\bullet\right)_\phi\right).$$

Definition 2.2.2 *The arithmetic class $\chi(P^\bullet, p_\bullet)$ of $(P^\bullet, p_\bullet)$ is defined to be that class in $A(\mathbf{Z}G)$ represented by the homomorphism on R_G which maps each $\phi \in \widehat{G}$ to the following element of $J_f \times \mathbf{R}_{>0}$:*

$$\prod_p \left(\prod_i \mathrm{Det}(\lambda_p^i)(\phi)^{(-1)^i}\right) \quad \times \quad p_\phi\left(\xi_\phi\left(\otimes_i b_{i,\phi}^{(-1)^i}\right)\right)^{\frac{1}{\phi(1)}}.$$

The following result from [16] will be crucial in subsequent applications:

Theorem 2.2.3 *Let $(P^\bullet, h_\bullet), (Q^\bullet, j_\bullet)$ be metrized perfect $\mathbf{Z}G$-complexes and let $f : P^\bullet \to Q^\bullet$ be a quasi-isomorphism, that is to say a chain map which induces an isomorphism on cohomology. Suppose further that f induces an isometry on their equivariant determinants of cohomology. Then*

$$\chi\left(P^\bullet, h_\bullet\right) = \chi\left(Q^\bullet, j_\bullet\right) \quad in \quad A\left(\mathbf{Z}G\right) \tag{2.2.4}$$

2.3 Rational classes and the symplectic arithmetic classgroup.

Let R_G^s denote the group of virtual symplectic characters of G. The symplectic arithmetic classgroup $A^s(\mathbf{Z}G)$ is defined in [16] to be the largest quotient of

$$\mathrm{Hom}_\Omega\left(R_G^s, J(\overline{\mathbf{Q}})^+\right) = \mathrm{Hom}_\Omega\left(R_G^s, J_f(\overline{\mathbf{Q}})\right) \times \mathrm{Hom}\left(R_G^s, \mathbf{R}_{>0}\right).$$

such that restriction of functions on R_G to R_G^s induces a homomorphism $A(\mathbf{Z}G) \to A^s(\mathbf{Z}G)$. In [16] we use the results of [7] to exhibit a subgroup $\mathcal{R}^s\left(\mathbf{Z}G\right)$ of $A^s\left(\mathbf{Z}G\right)$ which carries a natural discriminant isomorphism

$$\theta : \mathcal{R}^s\left(\mathbf{Z}G\right) \to \mathrm{Hom}_\Omega\left(R_G^s, \mathbf{Q}^\times\right) \tag{2.3.1}$$

We call $\mathcal{R}^s\left(\mathbf{Z}G\right)$ the subgroup of rational symplectic classes.

One can describe the inverse of θ in the following way. Let

$$\Delta_f : \mathbf{Q}^\times \to J(\overline{\mathbf{Q}})^+ = J_f(\overline{\mathbf{Q}}) \times \mathbf{R}_{>0} \tag{2.3.2}$$

be the embedding which is the diagonal map on each $r \in \mathbf{Q}_{>0}$, and for which $\Delta_f(-1) = (-1)_f$ is the idele in $J(\mathbf{Q})$ whose finite components equal -1 and whose infinite component is 1. Then Δ_f induces an injection

$$\Delta_{f,*} : \mathrm{Hom}_\Omega(R_G^s, \mathbf{Q}^\times) \to \mathrm{Hom}_\Omega\left(R_G^s, J(\overline{\mathbf{Q}})^+\right)$$

which gives rise to the inverse of θ when one identifies $A^s(\mathbf{Z}G)$ with a quotient of $\mathrm{Hom}_\Omega\left(R_G^s, J(\overline{\mathbf{Q}})^+\right)$.

2.4 Arakelov-Euler characteristics

The basic object of study here is a hermitian $G - \mathcal{X}$-bundle $(\mathcal{F}, j)$, which is defined to be a pair consisting of a locally free $\mathcal{O}_\mathcal{X}$ sheaf $\mathcal{F}$ having a G-action which is compatible with the action of G on $\mathcal{O}_\mathcal{X}$, and a G-invariant hermitian form j on the complex fibre $\mathcal{F}_\mathbf{C}$ of $\mathcal{F}$ which is invariant under complex conjugation.

From [9, 10] we know that since G acts tamely on $\mathcal{X}$, the $\mathbf{Z}G$-complex $R\Gamma(\mathcal{X}, \mathcal{F})$ is quasi-isomorphic to a perfect complex $P^\bullet$ in the derived category of $\mathbf{Z}G$ modules. We further recall that Quillen and Bismut have shown [4] how j induces so-called Quillen metrics $j_{Q,\bullet} = \{j_{Q,\phi}\}_{\phi \in \widehat{G}}$ on the equivariant determinants of the cohomology of $\mathcal{F}$.

We define the Arakelov-Euler characteristic of the hermitian $G-\mathcal{X}$ bundle $(\mathcal{F}, j)$ to be the class $\chi(P^\bullet, j_{Q,\bullet})$ in $A(\mathbf{Z}G)$. Note that this class is independent of choices by Theorem 2.2.3, so we will denote it by $\chi(R\Gamma(\mathcal{X}, \mathcal{F}), j_Q)$. We may extend the definition of $\chi(R\Gamma(\mathcal{X}, \mathcal{F}), j_Q)$ in a natural way to arguments $\mathcal{F}$ which are bounded complexes of hermitian G-modules. We will denote the image of $\chi(R\Gamma(\mathcal{X}, \mathcal{F}), j_Q)$ in the symplectic arithmetic class group $A^s(\mathbf{Z}G)$ by $\chi^s(R\Gamma(\mathcal{X}, \mathcal{F}), j_Q)$

The following technical hypothesis will be useful later.

Hypothesis 2.4.1 *The quotient scheme $\mathcal{Y} = \mathcal{X}/G$ is regular. The reductions of the finite fibers of $\mathcal{Y}$ have strictly normal crossings. The irreducible components of the finite fibers of $\mathcal{Y}$ have multiplicity prime to the residue characteristic.*

Let S denote a finite set of prime numbers which contains all the primes which support the branch locus of $\mathcal{X} \to \mathcal{Y} = \mathcal{X}/G$, together with all primes p where the fibre $\mathcal{Y}_p$ fails to be smooth. Define $\mathcal{X}_S^{\mathrm{red}}$ to be the disjoint union of the reduced fibers of $\mathcal{X}$ over the primes in S. Let $\Omega^1_\mathcal{X}(\log) =$

$\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log \mathcal{X}_S^{\mathrm{red}}/\log S)$ denote the sheaf of degree one relative logarithmic differentials with respect to the morphism $(\mathcal{X}, \mathcal{X}_S^{\mathrm{red}}) \to (\mathrm{Spec}(\mathbf{Z}), S)$ of schemes with log-structures (see [27]). Under hypotheses 2.4.1, $\Omega^1_{\mathcal{X}}(\log)$ is a locally free sheaf of rank d on $\mathcal{X}$ which restricts to $\Omega^1_{X/\mathbf{Q}}$ on the general fiber X of $\mathcal{X}$. For $i \geq 0$, let $\wedge^i h^D$ be the metric on $\wedge^i \Omega^1_{\mathcal{X}}(\log)$ which results from the Kähler metric h. Then $(\wedge^i \Omega^1_{\mathcal{X}}(\log), \wedge^i h^D)$ is a hermitian $G - \mathcal{X}$-bundle.

Definition 2.4.2 *Assume hypothesis 2.4.1. The log de Rham Euler characteristic of $\mathcal{X}$ with respect to S is*

$$\begin{aligned} \chi_{dRl}(\mathcal{X}, G, S) &= \chi(\mathrm{R}\Gamma(\wedge^\bullet \Omega^1_{\mathcal{X}}(\log), \wedge^\bullet h^D_Q) \\ &= \prod_{i=0}^{d} \chi(\mathrm{R}\Gamma(\wedge^i \Omega^1_{\mathcal{X}}(\log), \wedge^i h^D_Q)^{(-1)^i} \end{aligned} \tag{2.4.1}$$

where $\wedge^i h^D_Q$ denotes the Quillen metrics on the determinants of the isotypic parts of the cohomology of $\wedge^i \Omega^1_{\mathcal{X}}(\log)$. The image $\chi^s_{dRl}(\mathcal{X}, G, S)$ of $\chi_{dRl}(\mathcal{X}, G, S)$ in the symplectic arithmetic class group $A^s(\mathbf{Z}G)$ will be called the symplectic log de Rham Euler characteristic of $\mathcal{X}$.

2.5 Euler characteristics and Epsilon factors

Our goal is to relate the log de Rham Euler characteristic $\chi^s_{dRl}(\mathcal{X}, G, S)$ of definition 2.4.2 to the ϵ_0-factors associated to symplectic representations of G.

Let $\chi(\mathcal{Y}_{\mathbf{Q}}) = \chi(\mathcal{Y}(\mathbf{C}))$ denote the Euler characteristic of the generic fibre of $\mathcal{Y}$. Since $\mathcal{X} \to \mathcal{Y}$ is finite, the relative dimension of $\mathcal{Y}$ over $\mathrm{Spec}(\mathbf{Z})$ is d. Note that in all cases $d \cdot \chi(\mathcal{Y}_{\mathbf{Q}})$ is an even integer, so that we may define $\xi_S : R_G \to \mathbf{Q}^\times$ by the rule

$$\xi_S(\theta) = \prod_{p \in S} p^{-\theta(1) \cdot d \cdot \chi(\mathcal{Y}_{\mathbf{Q}})/2} \tag{2.5.1}$$

Let $\xi^s_S : R^s_G \to \mathbf{Q}^\times$ be the restriction of ξ_S to the group R^s_G of symplectic characters of G.

We need to introduce some notation for ϵ_0-constants. For a more detailed account see [12, §4] and [14, §2, 5]. For a given prime number p, we choose a prime number $l = l_p$ which is different from p and we fix a field embedding $\mathbf{Q}_l \to \mathbf{C}$. Following the procedure of [17, §8], each of the étale cohomology groups $\mathrm{H}^i_{\acute{e}t}(\mathcal{X} \times \bar{\mathbf{Q}}_p, \mathbf{Q}_l)$ for $0 \leq i \leq 2d$, affords a continuous complex representation of the local Weil-Deligne group. Thus, after choosing both an additive character ψ_p of $\mathbf{Q}_p$ and a Haar measure dx_p of $\mathbf{Q}_p$, for each complex character θ of G the complex number $\epsilon_{0,p}(\mathcal{Y}, \theta, \psi_p, dx_p, l_p)$ is defined. (For a representation V of G with character θ this term was denoted $\epsilon_{p,0}(\mathcal{X} \otimes_G V, \psi_p, dx_p, l)$ in [12, §2.4].) Setting $\widetilde{\epsilon}_{0,p}(\mathcal{Y}, \theta, \psi_p, dx_p, l_p) = \epsilon_{0,p}(\mathcal{Y}, \theta - \theta(1) \cdot 1, \psi_p, dx_p, l_p)$, by Corollary 1 to Theorem 1 in [13] we know that when θ is symplectic, $\widetilde{\epsilon}_{0,p}(\mathcal{Y}, \theta, \psi_p, dx_p, l_p)$ is a non-zero rational number, which is independent of choices, and $\theta \mapsto \widetilde{\epsilon}_{0,p}(\mathcal{Y}, \theta)$ defines an element

$$\widetilde{\epsilon}^s_{0,p}(\mathcal{Y}) \in \mathrm{Hom}_\Omega(R^s_G, \mathbf{Q}^\times) \tag{2.5.2}$$

Analogously, for the Archimedean prime ∞ of $\mathbf{Q}$, Deligne provides a definition for $\epsilon_\infty(\mathcal{Y})$, and from 5.5.2 and 5.4.1 in [12] we recall that

$$\widetilde{\epsilon}^s_\infty(\mathcal{Y}) \in \mathrm{Hom}_\Omega(R^s_G, \pm 1).$$

For $\phi \in R^s_G$ almost all $\widetilde{\epsilon}^s_{0,v}(\mathcal{Y}, \phi)$ are equal to 1. The global $\widetilde{\epsilon}_0$-constant of ϕ is

$$\widetilde{\epsilon}^s_0(\mathcal{Y}, \phi) = \prod_v \widetilde{\epsilon}^s_{0,v}(\mathcal{Y}, \phi)$$

and we define

$$\epsilon^s_{0,S}(\mathcal{Y}, \phi) = \widetilde{\epsilon}^s_0(\mathcal{Y}, \phi) \prod_{v \in S'} \epsilon_{0,v}(\mathcal{Y}, \phi(1)) \tag{2.5.3}$$

where $S' = S \cup \{\infty\}$. The main result proved in [16] concerning the log de Rham Arakelov Euler characteristic is:

Theorem 2.5.1 *The arithmetic class $\chi^s_{dRl}(\mathcal{X}, G, S)$ lies in the group of rational symplectic classes $R^s(\mathbf{Z}G)$ and*

$$\theta(\chi^s_{dRl}(\mathcal{X}, G, S)) = \xi^s_S \cdot \epsilon^s_{0,S}(\mathcal{Y})^{-1} \tag{2.5.4}$$

We refer the reader to [16] for an analogous result about a metrized Euler characteristic associated to the de Rham complex itself, or rather to a canonical complex of coherent G-sheaves on $\mathcal{X}$ which results from applying a construction of derived exterior powers due to Dold and Puppe to the relative differentials $\Omega^1_{\mathcal{X}/\mathbf{Z}}$. In [16] it is shown that the Arakelov Euler characteristic of this complex, together with a 'ramification class' associated to the bad fibers of $\mathcal{X}$, determines the symplectic ϵ-constants of $\mathcal{X}$.

2.6 A Conjecture of Bloch

In this section we let G be the trivial group, and we suppose initially only that $\mathcal{X}$ is a regular flat projective scheme over $\mathbf{Z}$ which is equidimensional of relative dimension d. Denote by $\mathcal{X}_S$ the disjoint union of the singular fibers of $f : \mathcal{X} \to \mathrm{Spec}(\mathbf{Z})$. For a zero cycle $\sum_i n_i x_i$, define

$$\mathrm{ord}(\sum_i n_i x_i) = \prod_i (\# k(x_i))^{n_i}$$

with $k(x_i)$ the residue field of x_i. For the definition of the conductor $A(\mathcal{X})$ of $\mathcal{X}$, see [5]. This definition requires the choice of an auxiliary prime l, in order to use Galois representations provided by the l-adic étale cohomology of the base change of the general fiber of $\mathcal{X}$ to $\overline{\mathbf{Q}}$. We will suppress dependence of l in the notation $A(\mathcal{X})$.

Conjecture 2.6.1 (Bloch [5]) *The conductor $A(\mathcal{X})$ is given by*

$$A(\mathcal{X}) = \mathrm{ord}((-1)^d c_{d+1}^{\mathcal{X}_S}(\Omega^1_{\mathcal{X}/\mathbf{Z}})) \tag{2.6.1}$$

where $c_{d+1}^{\mathcal{X}_S}(\Omega^1_{\mathcal{X}/\mathbf{Z}}) := c^{\mathcal{X}}_{d+1,\mathcal{X}_S}(\Omega^1_{\mathcal{X}/\mathbf{Z}}) \cap [\mathcal{X}]$ is the localized $d+1$-st Chern class in $\mathrm{CH}_0(\mathcal{X}_S)$ described in loc. cit.

We now describe a result from [15] which shows the equivalence of this conjecture with one concerning a metrized Euler characteristic.

As in the previous section, we choose a Kähler metric h on the tangent bundle of $\mathcal{X}(\mathbf{C})$, and we let h^D be the resulting hermitian metric on $\Omega^1_{\mathcal{X}(\mathbf{C})}$. Recall the definition of the arithmetic Grothendieck group $\widehat{\mathrm{K}}_0(\mathcal{X})$ of hermitian vector bundles of Gillet and Soulé ([22, §6]); all hermitian metrics are smooth and invariant under the complex conjugation on $\mathcal{X}(\mathbf{C})$. There is an arithmetic Euler characteristic homomorphism

$$\chi_Q : \widehat{\mathrm{K}}_0(\mathcal{X}) \longrightarrow \mathbf{R}$$

such that if $(\mathcal{F}, h)$ is a vector bundle on $\mathcal{X}$ with a hermitian metric on $\mathcal{F}_{\mathbf{C}}$, then $\chi_Q((\mathcal{F}, h))$ is the logarithmic Arakelov degree of the hermitian line bundle on $\mathrm{Spec}(\mathbf{Z})$ formed by the determinant of the cohomology of $\mathcal{F}$ with its Quillen metric. (Note that the Euler characteristic in (2.2.1) lies in the multiplicative group of positive real numbers, so on composing with the logarithm one has an additive Euler characteristic in $\mathbf{R}$.)

By work of Roessler in [32], the arithmetic Grothendieck group $\widehat{\mathrm{K}}_0(\mathcal{X})$ is a special λ-ring with λ^i-operations defined in [22, §7]: If $(\mathcal{F}, h)$ is the class of a vector bundle with a hermitian metric on $\mathcal{F}_{\mathbf{C}}$ then $\lambda^i((\mathcal{F}, h))$ is the class of the vector bundle $\wedge^i\mathcal{F}$ with the exterior power metric on $\wedge^i\mathcal{F}_{\mathbf{C}}$ induced from h. Now consider the sheaf of differentials $\Omega^1_{\mathcal{X}/\mathbf{Z}}$; this is a "hermitian coherent sheaf" in the terminology of [24, §2.5]. Since $\mathcal{X}$ is regular, by loc. cit. 2.5.2, $\Omega^1_{\mathcal{X}/\mathbf{Z}}$ defines an element Ω in $\widehat{\mathrm{K}}_0(\mathcal{X})$ as follows: Each embedding of $\mathcal{X}$ into projective space over $\mathrm{Spec}(\mathbf{Z})$ gives a short exact sequence

$$\mathcal{E} \;:\; 0 \;\to\; N \;\to\; P \;\to\; \Omega^1_{\mathcal{X}/\mathbf{Z}} \;\to\; 0 \tag{2.6.2}$$

with P and N vector bundles on $\mathcal{X}$ (here P is the restriction of the relative differentials of the projective space to $\mathcal{X}$ and N is the conormal bundle of the embedding). Pick hermitian metrics h^P and h^N on $P_{\mathbf{C}}$ and $N_{\mathbf{C}}$ respectively and denote by $\widetilde{\mathrm{ch}}(\mathcal{E}_{\mathbf{C}})$ the secondary Bott-Chern characteristic class of the exact sequence of hermitian vector bundles $\mathcal{E}_{\mathbf{C}}$ (as defined in [22]; there is a difference of a sign between this definition and the definition in [24, §2.5.2]). Then

$$\Omega = ((P, h^P), 0) - ((N, h^N), 0) + ((0,0), \widetilde{\mathrm{ch}}(\mathcal{E}_{\mathbf{C}})) \in \widehat{\mathrm{K}}_0(\mathcal{X})$$

depends only on the original choice of Kähler metric.

For each $i \geq 0$ we can consider now the element $\lambda^i(\Omega)$ in $\widehat{\mathrm{K}}_0(\mathcal{X})$.

Conjecture 2.6.2 *([15]) One has*

$$-\log |A(\mathcal{X})^{\frac{d+1}{2}}| = \sum_{i=0}^{d} (-1)^i \chi_Q(\lambda^i(\Omega)) \tag{2.6.3}$$

The following two results are proved in [15]

Theorem 2.6.3 *Conjecture 2.6.2 is equivalent to Bloch's conjecture 2.6.1.*

Theorem 2.6.4 *Bloch's conjecture 2.6.1, and therefore Conjecture 2.6.2, holds when for all primes p, the fiber of $\mathcal{X} \to \mathrm{Spec}(\mathbf{Z})$ over p is a divisor with strict normal crossings with multiplicities relative prime to p.*

The connection of Theorem 2.6.4 to results in §2.5 is that one can reformulate Bloch's conjecture in terms of the ϵ factor $\epsilon(\mathcal{X})$ of $\mathcal{X}$. As with $A(\mathcal{X})$, the definition of $\epsilon(\mathcal{X})$ requires the choice of an auxiliary prime l. One has

$$\epsilon(\mathcal{X})^2 = A(\mathcal{X})^{d+1} \tag{2.6.4}$$

Note that it is the ϵ-factor of $\mathcal{X}$ which occurs in (2.6.4) rather than the ϵ_0 factor. For this reason, Theorem 2.6.4 is equivalent to [16, Thm. 8.3] for the action of the trivial group G on $\mathcal{X}$, rather than to Theorem 2.5.1.

In [5], Bloch proved 2.6.1 when $d = 1$, i.e. for arithmetic surfaces. A result equivalent to Theorem 2.6.4 was proved independently by Arai in his thesis. Kato and Saito have proved 2.6.1 without the assumption that the multiplicities of the irreducible components of the fibers of $\mathcal{X}$ are prime to the residue characteristic (to appear). One other result related to these developments is an Arakelov-theoretic proof by Univer in [37] of a result of Saito [33] concerning the conductors and de Rham discriminants of arithmetic surfaces.

2.7 Sketch of the proofs

2.7.1 The proof of Theorem 2.6.3.

Denote by $\widehat{\mathrm{CH}}^{\cdot}(\mathcal{X})$, $\widehat{\mathrm{CH}}_{\cdot}(\mathcal{X})$ the arithmetic Chow groups of Gillet and Soulé ([22, 23]). The direct image homomorphism

$$f_* : \widehat{\mathrm{CH}}^{d+1}(\mathcal{X}) \to \widehat{\mathrm{CH}}^{1}(\mathrm{Spec}(\mathbf{Z})) = \mathbf{R}$$

satisfies $f_*(z_S(a)) = \log(\mathrm{ord}(a))$ for $a \in \mathrm{CH}_0(\mathcal{X}_S)$, where

$$z_S : \mathrm{CH}_0(\mathcal{X}_S) \to \widehat{\mathrm{CH}}_0(\mathcal{X}) = \widehat{\mathrm{CH}}^{d+1}(\mathcal{X})$$

is the natural homomorphism. Therefore, Theorem 2.6.3 is equivalent to

$$\sum_{i=0}^{d}(-1)^i \chi_Q(\lambda^i(\Omega)) = (-1)^{d+1}\frac{d+1}{2} f_*(z_S(c_{d+1}^{\mathcal{X}_S}(\Omega^1_{\mathcal{X}/\mathbf{Z}}))) \tag{2.7.1}$$

From the Arithmetic Riemann Roch theorem of Gillet and Soulé ([24, Theorem 7], see also 4.1.5 loc. cit.) we have

$$\begin{aligned}\sum_{i=0}^{d}(-1)^i \chi_Q(\lambda^i(\Omega)) \quad = \quad & f_*((\widehat{\mathrm{ch}}(\sum_{i=0}^{d}(-1)^i \lambda^i(\Omega)) \cdot \widehat{\mathrm{Td}}(\mathcal{X}))^{(d+1)}) \\ & -\frac{1}{2}\int_{\mathcal{X}(\mathbf{C})} \mathrm{ch}(\sum_{i=0}^{d}(-1)^i \Omega^i_{\mathcal{X}(\mathbf{C})})\mathrm{Td}(T_{\mathcal{X}(\mathbf{C})})R(T_{\mathcal{X}(\mathbf{C})})\end{aligned} \tag{2.7.2}$$

where the notations are as in loc. cit. and the factor of 1/2 in front of the second term results from the normalization discussed after equation (15) in section 4.1.5.

An argument of Soulé (c.f. [15, Prop. 2.3]) shows that the integral on the right in (2.7.2) is 0. This argument identifies the integrand with $c_d(T_{\mathcal{X}(\mathbf{C})})R(T_{\mathcal{X}(\mathbf{C})})$ via a classical Chern identity, and this product is 0 by considering the degree filtration of $\widehat{\mathrm{CH}}^{\cdot}(\mathcal{X})$.

A Chern class calculation (c.f. [16, Prop. 2.4]) shows

$$(\widehat{\mathrm{ch}}(\sum_{i=0}^{d}(-1)^i\lambda^i(\Omega))\cdot\widehat{\mathrm{Td}}(\mathcal{X})))^{(d+1)} = (-1)^{d+1}\frac{d+1}{2}\hat{c}_{d+1}(\Omega) \qquad (2.7.3)$$

These results and (2.7.2) show that to prove (2.7.1), and thus Theorem 2.6.3, it is enough to prove

$$z_S(c_{d+1}^{\mathcal{X}_S}(\Omega^1_{\mathcal{X}/\mathbf{Z}})) = \hat{c}_{d+1}(\Omega) \qquad (2.7.4)$$

The proof of (2.7.4) involves the construction of the localized Chern class via the Grassmanian graph construction (as described in [5, §1] or in [24, §1]) applied to the complex $0 \to N \to P$ with cokernel $\Omega^1_{\mathcal{X}/\mathbf{Z}}$ arising from (2.6.2). One defines a class $\mu \in \widehat{\mathrm{CH}}^{d+1}(\mathcal{X}\times\mathbf{P}^1)\otimes_{\mathbf{Z}}\mathbf{Q}$ via this construction whose restrictions to $\mathcal{X}\times\{0\}$ and $\mathcal{X}\times\{\infty\}$ are $\hat{c}_{d+1}(\Omega)$ and $z_S(c_{d+1}^{\mathcal{X}_S}(\Omega^1_{\mathcal{X}/\mathbf{Z}}))$, respectively. One then shows that these restrictions are equal by using [23, Theorem 4.4.6] to identify their difference with a (d,d) form on $\mathcal{X}(\mathbf{C})$ which can be shown by explicit computation to be 0. We refer the reader to [16, Lemmas 3.2 and 3.3] for further details.

2.7.2 The proof of Theorem 2.6.4.

With the notation and assumptions of §2.6, the exact sequence

$$0 \ \to\ N \ \to\ P \ \to\ \Omega^1_{\mathcal{X}/\mathbf{Z}} \ \to\ 0$$

together with the natural homomorphism

$$\Omega^1_{\mathcal{X}/\mathbf{Z}} \to \Omega^1_{\mathcal{X}/\mathbf{Z}}(\log) = \Omega^1_{\mathcal{X}/\mathbf{Z}}(\log\mathcal{X}_S^{\mathrm{red}}/\log S)$$

give a complex

$$\mathcal{E}_1 : N \ \stackrel{\delta}{\to}\ P \to \Omega^1_{\mathcal{X}/\mathbf{Z}}(\log) \qquad (2.7.5)$$

which is exact off S and concentrated in degrees -1, 0 and 1. We also have a complex

$$\mathcal{E}_2 : N \stackrel{(\delta,0)}{\to} P\oplus\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log) \ \stackrel{pr}{\to}\ \Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)$$

concentrated in the same degrees. There is a short exact sequence of complexes

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \ \to \ \Omega^1_{\mathcal{X}/\mathbf{Z}}(\log) \to 0$$

where on the right end, $\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)$ is considered as a complex supported on degree 0. Therefore, [1, Prop. 1.4][A] (see also [5, Prop. 1.1]) implies that

$$c^{\mathcal{X}_S}_{d+1}(\Omega^1_{\mathcal{X}/\mathbf{Z}}) = \sum_{k+l=d+1} c_k(\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)) \cdot c^{\mathcal{X}_S}_l([\mathcal{E}_1]) \tag{2.7.6}$$

Let $q = \prod_{p\in S} p$, and let $\{T_i\}_{i\in I}$ be the set of irreducible components of singular fibers of $\mathcal{X} \to \mathrm{Spec}(\mathbf{Z})$. Using Hypothesis 2.4.1, one can prove (c.f. [15, Prop. 4.1]) that in the Grothendieck group $K_0^{\mathcal{X}_S}(\mathcal{X})$ of complexes of locally free $\mathcal{O}_\mathcal{X}$-sheaves which are exact off $\mathcal{X}_S$, $\mathcal{E}_1$ in (2.7.5) has the same class as the complex

$$\mathcal{E}_3 : \mathcal{O}_\mathcal{X} \overset{(q,-q)}{\to} \mathcal{O}_\mathcal{X} \oplus (\oplus_i \mathcal{O}_\mathcal{X}(-T_i)) \overset{(id,\iota)}{\to} \oplus_i \mathcal{O}_\mathcal{X}$$

in which ι is induced by the natural inclusions $\mathcal{O}_\mathcal{X}(-T_i) \to \mathcal{O}_\mathcal{X}$ and id is the identity map. Therefore we have

$$c^{\mathcal{X}_S}_l([\mathcal{E}_1]) = c^{\mathcal{X}_S}_l([\mathcal{E}_3]) = c^{\mathcal{X}_S}_l([\mathcal{O}_\mathcal{X}/q\mathcal{O}_\mathcal{X}] + \sum_i(-[\mathcal{O}_{T_i}])) \tag{2.7.7}$$

Substituting (2.7.7) into (2.7.6), one deduces (c.f. [15])

$$\begin{aligned} c^{\mathcal{X}_S}_{d+1}(\Omega^1_{\mathcal{X}/\mathbf{Z}}) \ &= \ \sum_{i\in I}(m_i-1)c_d(\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)_{|T_i}) + \\ &\quad + \sum_{J\subset I, |J|\geq 2} (-1)^{|J|} c_{d+1-|J|}(\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)_{|T_J}) \end{aligned} \tag{2.7.8}$$

where $T_J = \cap_{i\in J} T_i$ for each subset J of I.

The logarithmic structure $\log \mathcal{X}_p^{\mathrm{red}}|T_J$ on T_J obtained by restricting $(\mathcal{X}, \mathcal{X}_p^{\mathrm{red}})$ to T_J is isomorphic to the logarithmic structure on T_J defined by its divisor with strict normal crossings $\cup_{J\neq\subset J'} T_{J'}$. This leads to an equality

$$[\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)_{|T_J}] = [\Omega^1_{T_J/k}(\log \mathcal{X}_p^{\mathrm{red}}|T_J)] + (|J|-1)[\mathcal{O}_{T_J}]$$

in $\mathrm{K}_0(T_J)$ (see [15, Prop. 4.8]). By considering Chern roots, one finds from this that

$$c_{d+1-|J|}(\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)_{|T_J}) = c_{d+1-|J|}(\Omega^1_{T_J/k}(\log \mathcal{X}_p^{\mathrm{red}}|T_J)) \tag{2.7.9}$$

From [34, p. 402],

$$\deg(c_{d+1-|J|}(\Omega^1_{T_J/k}(\log \mathcal{X}_p^{\mathrm{red}}|T_J))) = (-1)^{d+1-|J|}\chi_c(T_J^*) \tag{2.7.10}$$

where $\chi_c(T_J^*)$ is the l-adic ($l \notin S$) Euler characteristic with compact supports of $T_J^* = T_J - \cup_{J \neq \subset J'} T_{J'}$.

Combining (2.7.8), (2.7.9) and (2.7.10) shows

$$\begin{aligned}\deg((-1)^{d+1}c_{d+1}^{\mathcal{X}_S}(\Omega^1_{\mathcal{X}/\mathbf{Z}})_{|\mathcal{X}_p}) &= -\sum_{i\in I_p}(m_i-1)\chi_c^*(T_i) + \sum_{J\subset I_p, |J|\geq 2}\chi_c^*(T_J)\\ &= -\sum_{i\in I_p} m_i\chi_c^*(T_i) + \chi(\mathcal{X}_p) \qquad (2.7.11)\end{aligned}$$

where I_p is the subset of I that corresponds to components over p. Hypothesis 2.4.1 implies that the ramification of $\mathcal{X}$ is tame, in the sense that the Swan conductor associated to $\mathcal{X}$ is trivial. We thus find from [34, Cor. 2, p. 407] that the far right side of (2.7.11) is the power of p appearing in the conductor $A(\mathcal{X})$, and this completes the proof of Theorem 2.6.4.

2.7.3 The proof of Theorem 2.5.1.

Suppose first that G is the trivial group. Theorem 2.5.1 can then be shown by techniques similar to those used in the proof of the non-equivariant result Theorem 2.6.4. An alternate proof when G is trivial due to Bismut, which exploits Serre Duality and was pointed out to us by C. Soulé, is given in [16, Theorem 7.8].

Given Theorem 2.5.1 when G is trivial, the proof for general G is reduced in [16, §7] to considering characters of degree 0. More precisely, suppose in the notation of §2.3 that $f \in \mathrm{Hom}_\Omega\left(R_G^s, J(\overline{\mathbf{Q}})^+\right)$ is a character function representing a class c in the symplectic arithmetic classgroup $A^s(\mathbf{Z}G)$. The character function $\tilde{f}$ defined by $\tilde{f}(\chi) = f(\chi - \dim(\chi)1_G)$ then defines a class $\tilde{c}$ in $A^s(\mathbf{Z}G)$. Using Theorem 2.5.1 when G is trivial, we can reduce the general case to proving

$$\theta(\tilde{\chi}^s_{dRl}(\mathcal{X},G,S)) = \tilde{\xi}^s_S \cdot \tilde{\epsilon}^s_{0,S}(\mathcal{Y})^{-1} \tag{2.7.12}$$

Note that by (2.5.1), $\tilde{\xi}^s_S = 1$.

The strategy used to show (2.7.12) is similar to the one used in [14]. One would like to reduce the identity (2.7.12) to the case of one-dimensional $\mathcal{X}$, to which the methods used in studying the metrized Galois structure of rings of integers can be applied.

The first step in carrying out this reduction is to show that after a 'harmless' finite base extension of $\mathbf{Z}$, the d-th Chern class $c^d(\Omega^1_{\mathcal{Y}/\mathbf{Z}}(\log \mathcal{Y}_S^{\mathrm{red}}/\log S))$ associated to the rank d vector bundle $\Omega^1_{\mathcal{Y}/\mathbf{Z}}(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)$ on $\mathcal{Y} = \mathcal{X}/G$ can be written in $G_0(\mathcal{Y})$ as a sum

$$\sum_i n_i[\mathcal{O}_{\mathcal{D}_i}] + F$$

in which each $n_i = \pm 1$, $\mathcal{D}_i$ is a horizontal irreducible one-dimensional subscheme of $\mathcal{Y}$ which intersects the reduced fibers of $\mathcal{Y}$ transversely, and F is a class which will contribute nothing to later Euler characteristic computations. Since $\pi : \mathcal{X} \to \mathcal{Y}$ is a log-étale morphism, $\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log \mathcal{X}_S^{\mathrm{red}}/\log S)$ is the pullback of $\Omega^1_{\mathcal{Y}/\mathbf{Z}}(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)$. Because we have reduced to considering characters of degree 0, a comparison of Quillen metrics on the determinants of cohomology leads to a formula for $\tilde{\chi}^s_{dRl}(\mathcal{X}, G, S)$ in terms of metrized equivariant Euler characteristics associated to the structure sheaves of the normalizations Z_i of the $\pi^{-1}(\mathcal{D}_i)$. The metrics at infinity one uses on $\mathcal{O}_{Z_i}$ are simply those coming from the standard archimedean absolute values on the function fields of the irreducible components of Z_i.

The methods of Fröhlich, Taylor and Cassou-Noguès may be applied to compute the metrized Euler characteristic of the $\mathcal{O}_{Z_i}$ in terms of ϵ_0-factors over D_i. Results of Saito [34] then enable one to relate these ϵ_0-factors to those appearing on the right side of (2.7.12). For further details, see [16].

3 Equivariant discriminants and duality.

3.1 Hermitian modules and Fröhlich's Hermitian classgroup.

Let G be a finite group. All G-modules we consider will be left G-modules. Denote by $\alpha \to \overline{\alpha}$ the anti-involution of $\overline{\mathbf{Q}}G$ which is $\overline{\mathbf{Q}}$-linear and sends each $g \in G$ to g^{-1}. Suppose L is a finitely generated $\mathbf{Q}G$-module. An Hermitian pairing on L is a $\mathbf{Q}$-pairing

$$\langle , \rangle_G : L \times L \to \mathbf{Q}G \tag{3.1.1}$$

which is $\mathbf{Q}G$-linear in the second variable, and for which

$$\langle m, m' \rangle_G = \overline{\langle m', m \rangle_G} \tag{3.1.2}$$

for $m, m' \in L$. Such pairings are in bijection with G-invariant $\mathbf{Q}$-bilinear forms

$$\langle , \rangle : L \times L \to \mathbf{Q} \tag{3.1.3}$$

via the formula

$$\langle m, m' \rangle_G = \sum_{g \in G} \langle m, gm' \rangle \; g^{-1} \tag{3.1.4}$$

In [19], a Hermitian $\mathbf{Z}G$-module is defined to be a pair $(M, \langle , \rangle_G)$ consisting of a finitely generated locally free $\mathbf{Z}G$-module M and a non-degenerate hermitian pairing $\langle , \rangle_G$ on $M_{\mathbf{Q}} = \mathbf{Q} \otimes_{\mathbf{Z}} M$.

In [19], Fröhlich defined a quotient $HCl(\mathbf{Z}G)$ of

$$\mathrm{Hom}_{\Omega}(R_G, J(\overline{\mathbf{Q}}^{\times})) \times \mathrm{Hom}_{\Omega}(R^s_G, \overline{\mathbf{Q}}^{\times}) \tag{3.1.5}$$

called the Hermitian classgroup of $\mathbf{Z}G$. He associated to a Hermitian $\mathbf{Z}G$-module $(M, \langle , \rangle_G)$ a discriminant $d(M, \langle , \rangle_G)$ in $HCl(\mathbf{Z}G)$, by giving a recipe analogous to that in §2.2 for an element of (3.1.5). We refer the reader to [19, 11] for details.

3.2 The Adelic Hermitian classgroup and rational classes

In [19], Fröhlich defines the adelic Hermitian classgroup to be the quotient

$$Ad\ HCl(\mathbf{Z}G) = \frac{\mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R^s_G, J(\overline{\mathbf{Q}}))}{\mathrm{Det}^s(U(\mathbf{Z}G))} \tag{3.2.1}$$

where $\mathrm{Det}^s(U(\mathbf{Z}G))$ is the subgroup of $\mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R^s_G, J(\overline{\mathbf{Q}}))$ formed by the restrictions to R^s_G of character functions which are determinants of elements of the unit idele group $U(\mathbf{Z}G)$ of $\mathbf{Z}G$. There is a natural homomorphism

$$HCl(\mathbf{Z}G) \to Ad\ HCl(\mathbf{Z}G)$$

induced by the homomorphism

$$\mathrm{Hom}_{\Omega}(R_G, J(\overline{\mathbf{Q}}^{\times})) \times \mathrm{Hom}_{\Omega}(R^s_G, \overline{\mathbf{Q}}^{\times}) \to \mathrm{Hom}_{\Omega}(R^s_G, J(\overline{\mathbf{Q}}))$$

defined by

$$(h, f) \to h^s \cdot f.$$

where h^s is the restriction of h to R^s_G.

The homomorphism $\Delta_f : \mathbf{Q}^{\times} \to J(\overline{\mathbf{Q}})$ defined in (2.3.2) induces an injection

$$\Delta_{f,*} : \mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R^s_G, \mathbf{Q}^{\times}) \to Ad\ HCl(\mathbf{Z}G)$$

As in §2.3, we will call the image of $\Delta_{f,*}$ the rational symplectic classes $\mathcal{R}^s_{Ad}(\mathbf{Z}G)$ in $Ad\ HCl(\mathbf{Z}G)$. We let

$$\theta : \mathcal{R}^s_{Ad}(\mathbf{Z}G) \to \mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R^s_G, \mathbf{Q}^{\times}) \tag{3.2.2}$$

be the isomorphism whose inverse is the map induced by $\Delta_{f,*}$.

3.3 Discriminants of complexes.

Definition 3.3.1 *Suppose $P^\bullet$ is a perfect complex of $\mathbf{Z}G$-modules. A perfect pairing $\langle,\rangle$ on the cohomology $H^\bullet(P^\bullet_\mathbf{Q})$ is defined to be a collection $\{\langle,\rangle_t\}_t$ of non-degenerate G-invariant pairings*

$$\langle,\rangle_t : H^t(P^\bullet_\mathbf{Q}) \times H^{-t}(P^\bullet_\mathbf{Q}) \to \mathbf{Q}$$

such that

$$\langle x, y\rangle_t = \langle y, x\rangle_{-t} \tag{3.3.1}$$

for all $t \in \mathbf{Z}$, $x \in H^t(P^\bullet_\mathbf{Q})$ and $y \in H^{-t}(P^\bullet_\mathbf{Q})$. The pair $(P^\bullet, \langle,\rangle)$ will be called a perfect Hermitian complex. It will be said to be quasi-isomorphic to another perfect Hermitian complex $(P'^\bullet, \langle,\rangle')$ if there is an isomorphism between $P^\bullet$ and $P'^\bullet$ in the derived category of the homotopy category of complexes of $\mathbf{Z}G$-modules which are bounded above which identifies $\langle,\rangle$ with $\langle,\rangle'$.

Suppose $M^\bullet$ is a complex of finitely generated $\mathbf{Q}G$-modules. Let $\widetilde{M^\bullet}$ be the complex which results from applying to $M^\bullet$ the functor $M \to \tilde{M} = \mathrm{Hom}_\mathbf{Q}(M, \mathbf{Q})$ of finitely generated $\mathbf{Q}G$-modules M. The i^{th} term of $\widetilde{M^\bullet}$ is $\widetilde{M^{-i}}$.

Let $\langle,\rangle$ be a perfect pairing on a perfect complex $P^\bullet$. The following results are proved in [11]. There is an acyclic perfect complex $K^\bullet$ of $\mathbf{Z}G$-modules so that when $S^\bullet = P^\bullet \oplus K^\bullet$, there is a G-isomorphism of complexes $\phi = \phi(\langle,\rangle) : S^\bullet_\mathbf{Q} \to \widetilde{S^\bullet_\mathbf{Q}}$ with the following properties.

i. For all integers t, the isomorphism in cohomology

$$H^t(S^\bullet_\mathbf{Q}) \to H^t(\widetilde{S^\bullet_\mathbf{Q}}) = \mathrm{Hom}_\mathbf{Q}(H^{-t}(S^\bullet_\mathbf{Q}), \mathbf{Q})$$

induced by ϕ is the one induced by the pairing $\langle,\rangle_t$ together with the natural isomorphism of $H^{\pm t}(S^\bullet_\mathbf{Q})$ with $H^{\pm t}(P^\bullet_\mathbf{Q})$.

ii. Define $S^{even} = \bigoplus_{i\ even} S^i$ and $S^{odd} = \bigoplus_{i\ odd} S^i$. Then ϕ gives non-degenerate symmetric G-invariant pairings

$$\langle,\rangle^{even}_S : S^{even}_\mathbf{Q} \times S^{even}_\mathbf{Q} \to \mathbf{Q} \quad \text{and} \quad \langle,\rangle^{odd}_S : S^{odd}_\mathbf{Q} \times S^{odd}_\mathbf{Q} \to \mathbf{Q}$$

We now let $\langle,\rangle^{even}_{S,G}$ and $\langle,\rangle^{odd}_{S,G}$ be the Hermitian pairings on S^{even} and S^{odd}, respectively, which are associated to $\langle,\rangle^{even}_S$ and $\langle,\rangle^{odd}_S$ via the formula (3.1.4).

Theorem 3.3.2 *The quotient*

$$d(P^\bullet, \langle,\rangle) = \frac{d(S^{even}, \langle,\rangle^{even}_{S,G})}{d(S^{odd}, \langle,\rangle^{odd}_{S,G})}$$

in $HCl(\mathbf{Z}G)$ depends only on the quasi-isomorphism class of $(P^\bullet, \langle,\rangle)$, and will be called the discriminant of $(P^\bullet, \langle,\rangle)$.

3.4 Hermitian log de Rham discriminants.

In this section we describe an arithmetic application of the results of §3.3. We suppose as in §1 that $\mathcal{X}$ is a regular projective scheme which is flat and equidimensional over $\mathbf{Z}$ of relative dimension d, and that G is a finite group acting tamely on $\mathcal{X}$. We will also suppose hypothesis 2.4.1, so the sheaf $\Omega^1_{\mathcal{X}}(\log) = \Omega^1_{\mathcal{X}/\mathbf{Z}}(\log \mathcal{X}^{\mathrm{red}}_S / \log S)$ of degree one relative logarithmic differentials is locally free of rank d.

Let $\lambda^\bullet(\Omega^1_{\mathcal{X}}(\log))[d]$ be the complex of locally free $\mathcal{O}_{\mathcal{X}}$-modules having $\wedge^i\Omega^1_{\mathcal{X}}(\log)$ in dimension $i-d$ for $0 \le i \le d$, the zero sheaf in other dimensions, and trivial boundary maps. We have already seen that the hypercohomology $H^\bullet(\mathcal{X}, \lambda^\bullet(\Omega^1_{\mathcal{X}}(\log))[d])$ is represented by a perfect complex $P^\bullet$ of $\mathbf{Z}G$-modules because we have assumed G acts tamely on $\mathcal{X}$. Let $X = \mathbf{Q} \otimes_{\mathbf{Z}} \mathcal{X}$ be the general fiber of $\mathcal{X}$. The restriction of $\Omega^1_{\mathcal{X}}(\log)$ to X is $\Omega^1_X = \Omega^1_{X/\mathbf{Q}}$. Hence by flat base change, we find that for all t,

$$H^t(P^\bullet)_{\mathbf{Q}} = \bigoplus_{i+p=t+d} H^i(X, \Omega^p_X) \qquad (3.4.1)$$

From [25, III,§7] we have canonical perfect G-invariant duality pairings

$$\langle,\rangle_{i,j} : H^i(X, \Omega^j_X) \times H^{d-i}(X, \Omega^{d-j}_X) \to \mathbf{Q}$$

for $0 \le i, j \le d$. Define

$$\langle,\rangle'_{i,j} : H^i(X, \Omega^j_X) \times H^{d-i}(X, \Omega^{d-j}_X) \to \mathbf{Q}$$

by

$$\langle x, y\rangle'_{i,j} = \langle y, x\rangle_{d-i,d-j} \qquad (3.4.2)$$

By comparing $\langle,\rangle_{i,j}$ to the intersection pairing on Betti-cohomology (c.f. [21]), we see that

$$\langle x, y\rangle'_{i,j} = (-1)^{(i+j)}\langle x, y\rangle_{i,j} \qquad (3.4.3)$$

since $i+j$ and $2d-(i+j)$ have the same parity.

Definition 3.4.1 *Define a G-invariant non-degenerate pairing*

$$\langle,\rangle_t : H^t(P^\bullet)_{\mathbf{Q}} \times H^{-t}(P^\bullet)_{\mathbf{Q}} \to \mathbf{Q}$$

in the following way. If $t < 0$, let

$$\langle,\rangle_t = \bigoplus_{i+j=t+d} \langle,\rangle_{i,j}$$

relative to the canonical direct sum decomposition in (3.4.1) for t and $-t$. If $t > 0$, define

$$\langle x, y\rangle_t = \langle y, x\rangle_{-t} = \bigoplus_{i+j=t+d} \langle,\rangle'_{i,j} = (-1)^{t+d} \cdot \bigoplus_{i+j=t+d} \langle,\rangle_{i,j}.$$

Finally, if $t = 0$, let

$$\langle,\rangle_0 = \bigoplus_{i<d/2} \langle,\rangle_{i,d-i} \oplus \langle,\rangle_{d/2,d/2} \bigoplus_{i>d/2} \langle,\rangle'_{i,d-i}.$$

where the term $\langle,\rangle_{d/2,d/2}$ appears only if d is even.

Note that if d is even, then $\langle,\rangle_{d/2,d/2} = \langle,\rangle'_{d/2,d/2}$ is a symmetric pairing on $H^{d/2}(X, \Omega_X^{d/2})$ because of (3.4.3). Thus

$$\langle x, y\rangle_t = \langle y, x\rangle_{-t}$$

for all t, $x \in H^t(P^\bullet)_{\mathbf{Q}})$ and $y \in H^{-t}(P^\bullet)_{\mathbf{Q}})$.

Definition 3.4.2 *Define the Hermitian logarithmic de Rham discriminant of $(\mathcal{X}, G)$ in $HCl(\mathbf{Z}G)$ by*

$$\chi_{Hl}(\mathcal{X}, G, S) = d(\lambda^\bullet(\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log \mathcal{X}_S^{\mathrm{red}}/\log S))[d], \langle,\rangle) = d(P^\bullet, \langle,\rangle) \quad (3.4.4)$$

Let $\chi^A_{Hl}(\mathcal{X}, G, S)$ be the image of $\chi_{Hl}(\mathcal{X}, G, S)$ in the adelic Hermitian classgroup $Ad\ HCl(\mathbf{Z}G)$.

The image of $\chi_{Hl}(\mathcal{X}, G, S)$ in the usual classgroup $Cl(\mathbf{Z}G)$ is

$$(-1)^d \sum_{i=0}^{d} \chi_G(\Omega^i_{\mathcal{X}/\mathbf{Z}}(\log \mathcal{X}_S^{\mathrm{red}}/\log S))$$

where $\chi_G(\mathcal{F}) \in Cl(\mathbf{Z}G)$ is the Euler characteristic in $Cl(\mathbf{Z}G)$ of a coherent G-sheaf $\mathcal{F}$ on $\mathcal{X}$ as defined in [9]. This image arises in the study of the de Rham invariant $\chi(\mathcal{X}, G) \in Cl(\mathbf{Z}G)$ considered in [9, 12, 14], but is not exactly equal to $\chi(\mathcal{X}, G)$, since the latter pertains to the de Rham complex rather than the log de Rham complex. With more work, one can define a canonical Hermitian class $\chi_H(\mathcal{X}, G) \in HCl(\mathbf{Z}G)$ whose image in $Cl(\mathbf{Z}G)$ is $\chi(\mathcal{X}, G)$; see [11].

We end this section by discussing the classical case in which $d = 0$ and $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_N)$ for a tame G-extension N/K of number fields. Then $\mathcal{O}_N$ is a projective $\mathbf{Z}G$-module by a result of Noether, and $\chi(\mathcal{X}, G)$ is the stable isomorphism class $(\mathcal{O}_N) \in Cl(\mathbf{Z}G)$. The classes $\chi_{Hl}(\mathcal{X}, G, S)$ and $\chi_H(\mathcal{X}, G, S)$ are both equal to the class in $HCl(\mathbf{Z}G)$ of the Hermitian G-module $(\mathcal{O}_N, \mathrm{Tr}_{N/\mathbf{Q}})$ defined by the trace form associated to $N/\mathbf{Q}$. Fröhlich's conjecture relating $(\mathcal{O}_N) \in Cl(\mathbf{Z}G)$ to root numbers was proved by Taylor [36]; the corresponding Hermitian conjecture concerning $\chi^A_{Hl}(\mathcal{X}, G, S)$ was proved by Cassou-Noguès and Taylor in [7].

3.5 The case of surfaces.

The following result is shown in [11]. Since symplectic characters have even dimension, we may define a homomorphism $B' \in \mathrm{Hom}_\Omega(R^s_G, \mathbf{Q}^\times)$ by $B'(\chi) = (-1)^{\dim(\chi)/2}$. The diagonal map $\mathbf{Q}^\times \to J(\overline{\mathbf{Q}})$ allows us to view B' as an element of $\mathrm{Hom}_\Omega(R^s_G, J(\overline{\mathbf{Q}}))$; let B be the image of this element in $Ad\ HCl(\mathbf{Z}G)$ under the surjection in (3.2.1). Note that B will not be a rational class, since we have used the diagonal embedding of $\mathbf{Q}^\times$ into $J(\overline{\mathbf{Q}})$ rather than the homomomorphism Δ_f of (2.3.2). Let ξ^s_S and $\epsilon^s_{0,S}(\mathcal{Y})$ be the elements of $\mathrm{Hom}_\Omega(R^s_G, \mathbf{Q}^\times)$ defined in §2.5

Theorem 3.5.1 *Suppose* $\dim(\mathcal{X}) = d + 1 = 2$ *and that* $\mathcal{X}$ *satisfies hypothesis 2.4.1. Then* $B^{\chi(\mathcal{Y}_\mathbf{Q})} \cdot \chi^A_{Hl}(\mathcal{X}, G, S)$ *lies in the group of rational classes* $\mathcal{R}^s_{Ad}(\mathbf{Z}G)$ *and*

$$\theta(B^{\chi(\mathcal{Y}_\mathbf{Q})} \cdot \chi^A_{Hl}(\mathcal{X}, G, S)) = \xi^s_S \cdot \epsilon^s_{0,S}(\mathcal{Y})^{-1} \tag{3.5.1}$$

where θ *is defined in (3.2.2) and* $\chi(\mathcal{Y}_\mathbf{Q})$ *is the Euler characteristic of the general fiber* $\mathcal{Y}_\mathbf{Q}$ *of* $\mathcal{Y}$.

It is somewhat mysterious that the same ϵ_0-factors arise in describing $\chi^A_{Hl}(\mathcal{X}, G, S)$ and the Arakelov theoretic class $\chi^s_{dRl}(\mathcal{X}, G, S)$ of Theorem 2.5.1. We see no direct reason for this to be true, e.g. because the metrics on cohomology in the Arakelov approach are positive definite while the pairings on cohomology used to define $\chi^A_{Hl}(\mathcal{X}, G, S)$ will be indefinite in general.

3.6 Sketch of the proof of Theorem 3.5.1.

When G is the trivial group and $\dim(\mathcal{X}) = 2$, Theorem 3.5.1 follows from work of Saito [34] on ϵ_0-factors and of Bloch [5] on the relation between conductors and de Rham discriminants. The factor $B^{\chi(\mathcal{Y}_{\mathbf{Q}})}$ arising in (3.5.1) comes the computation of the discriminants of hyperbolic Hermitian modules given in [19, Prop. II.5.7]. As in §2.7.3, knowing Theorem 3.5.1 when G is trivial reduces one to considering characters of degree 0 for general G.

To treat characters of degree 0, the strategy (as in [12]) is to reduce to the case of rings of integers by a judicious choice of effective divisors on $\mathcal{Y} = \mathcal{X}/G$. Suppose, for example, that one has a global section s of $\Omega^1_{\mathcal{Y}/\mathbf{Z}}(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)$, and that $\mathcal{C}$ is the (effective) divisor of s. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be the natural quotient map, and let $\mathcal{C}' = \pi^{-1}(\mathcal{C})$. Then one has an exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \xrightarrow{s} \Omega^1_{\mathcal{X}/\mathbf{Z}}(\log \mathcal{X}_S^{\mathrm{red}}/\log S) \to \mathcal{O}_{\mathcal{C}'} \to 0 \tag{3.6.1}$$

of coherent G-sheaves. The cohomology of this sequence can be used to relate the difference of the Euler characteristics of $\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log \mathcal{X}_S^{\mathrm{red}}/\log S)$ and $\mathcal{O}_{\mathcal{X}}$ to that of $\mathcal{O}_{\mathcal{C}'}$. To show such a relation for Hermitian Euler characteristics, one must compare the duality pairings on the cohomology of $\mathcal{O}_{\mathcal{X}}$ and $\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log \mathcal{X}_S^{\mathrm{red}}/\log S)$ to the trace pairing on the cohomology of $\mathcal{O}_{\mathcal{C}'}$.

Since one does not in general have a global section of $\Omega^1_{\mathcal{Y}/\mathbf{Z}}(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)$, a more involved argument is used in [11] in which one chooses two effective horizontal divisors $\mathcal{D}$ and $\mathcal{J}$ on $\mathcal{Y}$. These divisors have the property that

$$K_{\mathcal{Y}} + \mathcal{Y}_S^{red} + 2\mathcal{J} + \mathcal{F} = \mathcal{D}$$

for a canonical divisor $K_{\mathcal{Y}}$ on $\mathcal{Y}$, where $\mathcal{Y}_S^{red}$ is the sum of the reductions $\mathcal{Y}_p^{red}$ of the fibers of $\mathcal{Y}$ over the primes p in S, and $\mathcal{F}$ is a linear combination of irreducible components of fibers of $\mathcal{Y} \to \mathrm{Spec}(\mathbf{Z})$ over primes not in S. The algebraic problem is to show that

$$\tilde{\chi}^A_{Hl}(\mathcal{X}, G, S) = \tilde{d}(H^0(\mathcal{O}_{\mathcal{D}'}), tr_{D'}) \cdot \tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'}), tr_{J'})^{-2} \tag{3.6.2}$$

where $\tilde{c}$ is the restriction to characters of degree 0 of a class c, $\mathcal{D}' = \pi^{-1}(\mathcal{D})$, $\mathcal{J}' = \pi^{-1}(\mathcal{J})$ and $tr_{D'}$ and $tr_{J'}$ are the trace forms on the generic fibers D' and J' of $\mathcal{D}'$ and $\mathcal{J}'$, respectively. This requires establishing various exact sequences in cohomology and comparing via these sequences the pairings involved in the definition of the discriminants appearing in (3.6.2).

The proof of Theorem 3.5.1 is completed by using the main result of Cassou-Noguès and Taylor in [7] to relate the right hand side of (3.6.2) to root numbers associated to $\mathcal{D}'$ and $\mathcal{J}'$, and the comparison of these root numbers to those going into the definition of $\tilde{\epsilon}^s(\mathcal{Y})^{-1}$ which was made in [12, §9]. The latter comparison again relies on the formulas of Saito in [34]; for further details, see [11].

4 An example

Theorem 4.0.1 *Let F be a number field. Suppose Hypothesis 2.4.1 holds, and that $\mathcal{Y} = \mathcal{X}/G$ is a projective, flat, regular model over O_F of an elliptic curve over F having reduced fibers. Then the Euler characteristics $\chi^s_{dRl}(\mathcal{X}, G, S)$ and $\chi^A_{Hl}(\mathcal{X}, G, S)$ appearing in Theorems 2.5.1 and 3.5.1, respectively, are both trivial, as are the character functions ξ^s_S and $\epsilon^s_{0,S}(\mathcal{Y})$.*

We first indicate one way to construct examples in which the hypotheses of this Theorem are satisfied. Let $\mathcal{E}$ be a regular model over $\mathcal{O}_F$ of an elliptic curve over F, and suppose that the singular fibers of $\mathcal{E}$ are reduced and of multiplicative type. Suppose that there is a finite subgroup Γ of $\mathcal{E}(F)$ which maps injectively into each smooth fiber of $\mathcal{E}$ and injectively into the group of connected components of each singular fiber of $\mathcal{E}$. Then Γ gives a finite group of automorphisms of $\mathcal{E}$ such that the quotient map $\mathcal{E} \to \mathcal{E}/\Gamma = \mathcal{Y}$ is an étale Γ-cover. Let N/F be a finite tame Galois extension which is unramified over the places of F where $\mathcal{E}$ has bad reduction. The scheme $\mathcal{X} = \mathrm{Spec}(O_N) \times \mathcal{E}$ is a tame Galois cover of Y with group $\mathrm{Gal}(N/F) \times \Gamma = G$, and the hypothesis of Theorem 4.0.1 hold.

To prove Theorem 4.0.1, observe first that by [16, Lemma 7.10], $\Omega^1_{\mathcal{Y}/\mathbf{Z}}(\log \mathcal{Y}^{\mathrm{red}}_S / \log S)$ is naturally isomorphic to the relative dualizing sheaf $\omega_{\mathcal{Y}/\mathbf{Z}}$ since $\mathcal{Y}$ has reduced special fibers. By [3, Prop. 1.15], the Neron model $\mathcal{Y}^N$ of $\mathcal{Y}$ consists of the complement of the (codimension two) singular points in the singular fibers of $\mathcal{Y}$. Since $\Omega^1_{\mathcal{Y}^N/\mathbf{Z}}$ is the trivial bundle on $\mathcal{Y}^N$, it follows that $\omega_{\mathcal{Y}/\mathbf{Z}}$ is isomorphic to $\mathcal{O}_{\mathcal{Y}}$. We conclude on pulling back via $\pi : \mathcal{X} \to \mathcal{Y} = \mathcal{X}/G$ that $\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log) = \Omega^1_{\mathcal{X}/\mathbf{Z}}(\log \mathcal{X}^{\mathrm{red}}_S / \log S) = \pi^*\Omega^1_{\mathcal{Y}/\mathbf{Z}}(\log \mathcal{Y}^{\mathrm{red}}_S / \log S)$ is equivariantly isomorphic to $\mathcal{O}_{\mathcal{X}}$. By [16, Thm. 6.2], we therefore have an equality of degree 0 classes

$$\tilde{\chi}\left(R\Gamma(\mathcal{X}, \mathcal{O}_X), \Lambda^0 h^D_Q\right) = \tilde{\chi}\left(R\Gamma(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)), \Lambda^1 h^D_Q\right) \tag{4.0.1}$$

where $\Lambda^i h^D_Q$ is the Quillen metric in cohomology associated to a choice of

Kähler metric on $\mathcal{X}$ (see Definition 2.4.2). Therefore

$$\begin{aligned}\tilde{\chi}^s_{dRl}(\mathcal{X},G,S) &= \tilde{\chi}\left(R\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}}),\Lambda^0 h^D_Q\right)\cdot\tilde{\chi}\left(R\Gamma(\mathcal{X},\Omega^1_{\mathcal{X}/\mathbf{Z}}(\log)),\Lambda^1 h^D_Q\right)^{-1} \\ &= 1 \end{aligned} \qquad (4.0.2)$$

By Theorem 2.5.1,

$$\theta(\chi^s_{dRl}(\mathcal{X},G,S)) = \xi^s_S\cdot\epsilon^s_{0,S}(\mathcal{Y})^{-1} \qquad (4.0.3)$$

Since the generic fiber $\mathcal{Y}_{\mathbf{Q}}$ of $\mathcal{Y}$ has been assumed to be an elliptic curve, one has $\xi^s_S = 0$ from (2.5.1). Thus (4.0.2) and (4.0.3) show that $\theta(\chi^s_{dRl}(\mathcal{X},G,S))$ and $\epsilon^s_{0,S}(\mathcal{Y})$ are trivial on characters of degree 0. If χ is a virtual symplectic character of G, then $\dim(\chi)$ is even, and $\chi-\dim(\chi)\chi_0$ is a virtual symplectic character of degree 0, where χ_0 is the trivial character and $2\chi_0$ is symplectic. Since θ is injective (c.f. §2.3), we conclude that to show to $\chi^s_{dRl}(\mathcal{X},G,S)=0$, it will suffice to prove

$$\epsilon^s_{0,S}(\mathcal{Y})(2\chi_0) = 1 \qquad (4.0.4)$$

This equality follows from [16, Theorem 7.9] since all of the fibers of $\mathcal{Y}$ have been assumed to be reduced. Since we have now shown ξ^s_S and $\epsilon^s_{0,S}$ are both trivial, we have $\chi^A_{Hl}(\mathcal{X},G,S)$ by Theorem 3.5.1. This completes the proof of Theorem 4.0.1.

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