DEFORMATIONS AND DERIVED CATEGORIES

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1. Introduction.

Suppose k is a field of characteristic $p > 0$, W is a complete local commutative Noetherian ring with residue field k, and that G is a profinite group. In [19], Mazur developed a deformation theory of finite dimensional representations of G over k. His work was based on that of Schlessinger in [25]; a more explicit approach was later described by de Smit and Lenstra in [12]. Deformation theory has become a basic tool in arithmetic geometry (see e.g. [10], [28], [27], [7], and their references). In this paper we generalize the theory by considering instead of k-representations of G objects in the derived category $D^-([[kG]])$ of bounded above complexes of pseudocompact modules over the completed group algebra $[[kG]]$ of G over k. The case of k-representations amounts to studying complexes which have exactly one non-zero cohomology group.

We have two reasons for pursuing this generalization. The first is that objects in derived categories occur in a natural way in number theory and arithmetic geometry, and they have played an important role in deformation theory (see e.g. the work of Illusie [18]). Galois cohomology classes, for example, provide such objects when one takes the mapping cone of the associated morphism in the derived category. It is a natural problem to consider the deformations of cohomology classes in this way,

in view of the interest of deforming Galois modules and the fact that cohomology classes play a central role in many questions, e.g. in class field theory. Another way in which objects in derived categories arise is as the hypercohomology of sheaves in various topologies, e.g. the étale topology, on schemes. Such hypercohomology complexes often carry more information than their individual cohomology groups. Since it has been advantageous to find arithmetic constructions of universal deformations of various Galois modules, it seems a natural problem to look for the corresponding constructions for complexes of Galois modules

The second motivation for this paper arises from the study of universal deformations for finite groups. Conjectures of Broué and others (see e.g. [8], [24] and their references) would establish derived equivalences between various blocks of group rings of finite groups. In [2] and [5], Morita equivalences between module categories were a basic tool used in determining the universal deformation rings of representations associated to blocks with cyclic or Klein four defect groups. Because derived equivalences are conjectured to exist in a much broader context, it is natural to consider deformations of objects in derived categories.

This article is organized in the following way. In §2 we state our main result, Theorem 2.14, which extends Mazur's deformation theory to objects $V^\bullet$ in $D^-([[kG]])$. We assume that $V^\bullet$ has only finitely many cohomology groups, all of which have finite k-dimension. Mazur's finiteness condition (Φ_p) in [19, §1.1] for G is replaced by the condition that G has finite pseudocompact cohomology (see Definition 2.13). We consider two types of lifts: quasi-lifts and proflat quasi-lifts (see Definition 2.7). Theorem 2.14 states that $V^\bullet$ has a versal deformation ring (resp. a versal proflat deformation ring), and this ring is universal if the endomorphism ring of $V^\bullet$ in the derived category is equal to the scalar multiplications provided by k. We prove Theorem 2.14 in §2 through §7. In §8 we determine the behavior of versal deformation rings (resp. versal proflat deformation rings) under finite extensions of the residue field k.

In §9 we study complexes $V^\bullet$ which have either one or two non-zero cohomology groups. In Proposition 9.3 we give a necessary and sufficient condition for a complex with two non-zero cohomology groups to have endomorphism ring k in the derived category. In Example 9.5 we show that one can construct two-term complexes $V^\bullet$ with endomorphism ring k so that at least one of the cohomology groups of $V^\bullet$ has endomorphism ring larger than k. This shows that complexes can have universal deformation rings

even when one does not know that the individual cohomology groups of the complex have universal deformation rings. In Proposition 9.6 we determine the tangent space of the proflat deformation functor in case $V^\bullet$ has exactly two non-zero cohomology groups $U_0 = \mathrm{H}^0(V^\bullet)$ and $U_{-n} = \mathrm{H}^{-n}(V^\bullet)$ for some $n > 0$. If in addition U_{-n} and U_0 have universal deformation rings R_{-n} and R_0 in the sense of Mazur [19], we give in Propositions 9.7 and 9.8 criteria for when the versal proflat deformation ring of such a $V^\bullet$ is universal and isomorphic to $R_{-n} \widehat{\otimes}_W R_0$, where $\widehat{\otimes}_W$ denotes the completed tensor product over W.

In §10 we look at deformations of group cohomology elements. As an example, we consider the nontrivial element $\beta \in \mathrm{H}^2(G, k)$ where G is the absolute Galois group of $\mathbb{Q}_\ell$, $\ell > 2$ is a rational prime, $k = \mathbb{Z}/2$, and $W = \mathbb{Z}_2$. We determine the tangent space of the deformation functor associated to the mapping cone of β. We prove in Theorem 10.6 that the versal proflat deformation ring is universal and isomorphic to $[[WG^{\mathrm{ab},2}]]\widehat{\otimes}_W[[WG^{\mathrm{ab},2}]]$, where $[[WG^{\mathrm{ab},2}]]$ is the completed group ring over W of the abelianized 2-completion of G.

In §11, we consider the case in which $V^\bullet$ is a completely split complex, in the sense that it is isomorphic to a complex in $D^-([[kG]])$ having trivial boundary maps. We discuss in Proposition 11.3 a split deformation functor which we show is the proflat deformation functor of Theorem 2.14 when $V^\bullet$ is completely split.

In §12 we consider the étale hypercohomology of locally constant constructible sheaves $\mathcal{F}$ of k-vector spaces on an abelian variety X over $\mathbb{Q}$. We prove in Theorem 12.1 that if $p > 2\dim(X)$, then the étale hypercohomology $\mathrm{H}^\bullet(\overline{X}, \mathcal{F})$ of $\mathcal{F}$ is completely split in $D^-([[kG_{\mathbb{Q}}]])$, where $G_{\mathbb{Q}}$ is the absolute Galois group of $\mathbb{Q}$. The same is true if $p > 2\dim(X) - 2$, provided there exists a k-bilinear non-degenerate form $\mathcal{F} \times \mathcal{F} \to k(d)$, where $k(d)$ is the d-th Tate twist of the constant sheaf k. In these results, one can replace $\mathrm{H}^\bullet(\overline{X}, \mathcal{F})$ by the compact hypercohomology $\mathrm{H}_c^\bullet(\overline{U}, \mathcal{F})$ of $\mathcal{F}$ over the complement $\overline{U}$ of the origin in $\overline{X}$.

In §13, we specialize to the case in which X is an elliptic curve over $\mathbb{Q}$ with origin Q. Suppose S is the finite set of places of $\mathbb{Q}$ consisting of the archimedean place together with the finite places determined by p and the primes of bad reduction for X, and let G_S be the Galois group over $\mathbb{Q}$ of the maximal algebraic extension $\mathbb{Q}_S$ of $\mathbb{Q}$ unramified outside S. Because of the results in §12, we can regard $\mathrm{H}_c^\bullet(\overline{U}, \mu_p)$ as a completely split complex $V^\bullet$ in $D^-([[kG_S]])$ when $k = \mathbb{Z}/p$. In Theorem 13.10 we show that

for some CM elliptic curves X over $\mathbb{Q}$ studied by Boston and Ullom [6], the versal deformation of $V^\bullet$ is universal and completely split. Moreover, the universal deformation ring $R(G_S, V^\bullet)$ is isomorphic to a power series algebra in four commuting indeterminates over $W = \mathbb{Z}_p$.

In §14, we provide some results from Milne [21] which we have restated to fit our situation.

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Some of the results of this article have been announced in [4].

2. Quasi-lifts and deformation functors.

Let G be a profinite group, let k be a field of characteristic $p > 0$, and let W be a complete local commutative Noetherian ring with residue field k. Define $\widehat{\mathcal{C}}$ to be the category of complete local commutative Noetherian W-algebras with residue field k. The morphisms in $\widehat{\mathcal{C}}$ are continuous W-algebra homomorphisms which induce the identity on k. Let $\mathcal{C}$ be the subcategory of Artinian objects in $\widehat{\mathcal{C}}$. If $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$, let $[[RG]]$ be the completed group algebra of the usual abstract group algebra $[RG]$ of G over R, i.e. $[[RG]]$ is the projective limit of the ordinary group algebras $[R(G/U)]$ as U runs through the open normal subgroups of G.

DEFINITION 2.1. — A topological ring Λ is called a *pseudocompact ring* if Λ is complete and Hausdorff and admits a basis of open neighborhoods of 0 consisting of two-sided ideals J for which Λ/J is an Artinian ring.

Suppose Λ is a pseudocompact ring. A complete Hausdorff topological Λ-module M is said to be a *pseudocompact Λ-module* if M has a basis of open neighborhoods of 0 consisting of submodules N for which M/N has finite length as Λ-module. We denote by $\mathrm{PCMod}(\Lambda)$ the category of pseudocompact Λ-modules.

A pseudocompact Λ-module M is said to be *topologically free* on a set $X = \{x_i\}_{i \in I}$ if M is isomorphic to the product of a family $(\Lambda_i)_{i \in I}$ where $\Lambda_i = \Lambda$ for all i.

Suppose R is a commutative pseudocompact ring. A complete Hausdorff topological ring Λ is called a *pseudocompact R-algebra* if Λ is an R-algebra in the usual sense, and if Λ admits a basis of open neighborhoods of 0 consisting of two-sided ideals J for which Λ/J has finite length as R-module.

Suppose Λ is a pseudocompact R-algebra, and let $\widehat{\otimes}_\Lambda$ denote the completed tensor product in the category $\mathrm{PCMod}(\Lambda)$. A pseudocompact Λ-module M is said to be *topologically flat*, if the functor $M \widehat{\otimes}_\Lambda -$ is exact.

Remark 2.2. — Pseudocompact rings, algebras and modules have been studied, for example, in [13], [14], [9]. The following statements can be found in these references. Suppose Λ is a pseudocompact ring.

(i) The ring Λ is the projective limit of Artinian quotient rings having the discrete topology. A Λ-module is pseudocompact if and only if it is the projective limit of Λ-modules of finite length having the discrete topology. The category $\mathrm{PCMod}(\Lambda)$ is an abelian category with exact projective limits.

(ii) Every topologically free pseudocompact Λ-module is a projective object in $\mathrm{PCMod}(\Lambda)$, and every pseudocompact Λ-module is the quotient of a topologically free Λ-module. Hence $\mathrm{PCMod}(\Lambda)$ has enough projective objects.

(iii) Every pseudocompact R-algebra is a pseudocompact ring, and a module over a pseudocompact R-algebra has finite length if and only if it has finite length as R-module.

(iv) Suppose Λ is a pseudocompact R-algebra, and A and B are pseudocompact Λ-modules. Then we define the right derived functors $\mathrm{Ext}^n_\Lambda(A, B)$ by using a projective resolution of A.

(v) Suppose $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$. Then R is a pseudocompact ring, and $[[RG]]$ is a pseudocompact R-algebra. Note that if k is finite, then R and $[[RG]]$ are actually profinite rings, and pseudocompact $[[RG]]$-modules are the same as profinite $[[RG]]$-modules, which have been studied e.g. in [23].

Remark 2.3. — Let R be an object in $\widehat{\mathcal{C}}$ with maximal ideal m_R. Suppose that $[(R/m_R^i)X_i]$ is an abstractly free (R/m_R^i)-module on the finite topological space X_i for all i, and that $\{X_i\}_i$ forms an inverse system. Define $X = \varprojlim_i X_i$ and $[[RX]] = \varprojlim_i [(R/m_R^i)X_i]$. Then $[[RX]]$ is a topologically free pseudocompact R-module on X. In particular, every topologically free pseudocompact $[[RG]]$-module is a topologically free pseudocompact R-module.

Remark 2.4. — Suppose R is an object in $\widehat{\mathcal{C}}$ and M is a pseudocompact R-module. If M is finitely generated as a pseudocompact R-module, then the functors $M \otimes_R -$ and $M \widehat{\otimes}_R -$ are naturally isomorphic (see [9, Lemma 2.1 (i)]). For general pseudocompact R-modules M, it follows from [14, proof of Prop. 0.3.7] and [14, Cor. 0.3.8] that M is topologically flat if and only if M is topologically free if and only if M is abstractly flat. In particular, if R is Artinian, a pseudocompact R-module is topologically flat if and only if it is abstractly free.

Let $C^-([[RG]])$ be the abelian category of complexes of pseudocompact $[[RG]]$-modules which are bounded above, let $K^-([[RG]])$ be the homotopy category of $C^-([[RG]])$, and let $D^-([[RG]])$ be the derived category of $K^-([[RG]])$. Let T denote the translation functor on $D^-([[RG]])$ (resp. $K^-([[RG]])$, resp. $C^-([[RG]])$), i.e. T shifts complexes one place to the left and changes the sign of the differential.

DEFINITION 2.5. — We will say that a complex $M^\bullet$ in $K^-([[RG]])$ has *finite pseudocompact R-tor dimension*, if there exists an integer N such that for all pseudocompact R-modules S, and for all integers $i < N$, $\mathrm{H}^i(S \widehat{\otimes}_R^{\boldsymbol{L}} M^\bullet) = 0$. Note that $\widehat{\otimes}_R^{\boldsymbol{L}}$ stands for the left derived functor of $\widehat{\otimes}_R$. If we want to emphasize the integer N in this definition, we say $M^\bullet$ has *finite pseudocompact R-tor dimension at N*.

Remark 2.6. — Suppose that $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$ and that $M^\bullet$ is a complex in $K^-([[RG]])$ of topologically flat, hence topologically free, pseudocompact R-modules. Then $S \widehat{\otimes}_R^{\boldsymbol{L}} M^\bullet = S \widehat{\otimes}_R M^\bullet$ for all pseudocompact R-modules S; in particular $k \widehat{\otimes}_R^{\boldsymbol{L}} M^\bullet = k \widehat{\otimes}_R M^\bullet$. Suppose additionally that $M^\bullet$ has finite pseudocompact R-tor dimension at N. Then it follows (cf. [21, proof of Cor. VI.8.10]) that the bounded complex $M'^\bullet$, obtained from $M^\bullet$ by replacing M^N with $M'^N = M^N/\delta^{N-1}(M^{N-1})$ and by setting $M'^i = 0$ if $i < N$, is quasi-isomorphic to $M^\bullet$ and has topologically free pseudocompact terms over R.

HYPOTHESIS 1. — Throughout this paper, we assume that $V^\bullet$ is a complex in $D^-([[kG]])$ which has only finitely many non-zero cohomology groups, all of which have finite k-dimension.

DEFINITION 2.7. — A *quasi-lift* of $V^\bullet$ over an object R of $\widehat{\mathcal{C}}$ is a pair $(M^\bullet, \phi)$ consisting of a complex $M^\bullet$ in $D^-([[RG]])$ which has finite pseudocompact R-tor dimension together with an isomorphism

$\phi : k \widehat{\otimes}_R^{\boldsymbol{L}} M^\bullet \to V^\bullet$ in $D^-([[kG]])$. Two quasi-lifts $(M^\bullet, \phi)$ and $(M'^\bullet, \phi')$ are *isomorphic* if there is an isomorphism $M^\bullet \to M'^\bullet$ in $D^-([[RG]])$ which carries ϕ to ϕ'. A *deformation* of $V^\bullet$ over R is an isomorphism class of quasi-lifts of $V^\bullet$.

A *proflat quasi-lift* of $V^\bullet$ over an object R of $\widehat{\mathcal{C}}$ is a quasi-lift $(M^\bullet, \phi)$ of $V^\bullet$ over R whose cohomology groups are topologically flat, and hence topologically free, pseudocompact R-modules. A *proflat deformation* of $V^\bullet$ over R is an isomorphism class of proflat quasi-lifts of $V^\bullet$.

Remark 2.8. — There exist quasi-lifts which are not isomorphic to proflat quasi-lifts in $D^-([[RG]])$. For example, suppose k is a perfect field, $W = W(k)$ is the ring of infinite Witt vectors over k, G is the trivial group, and $V^\bullet = k \xrightarrow{0} k$ is the two-term complex concentrated in dimensions -1 and 0 with trivial boundary map. Let $R = W$, and let $M^\bullet = W \xrightarrow{p} W$ be the quasi-lift of $V^\bullet$ over W concentrated in dimensions -1 and 0 with boundary map given by multiplication by p. Then $M^\bullet$ is isomorphic to the one-term complex W/pW concentrated in dimension 0, but its single non-zero cohomology group is not a topologically free pseudocompact W-module.

LEMMA 2.9. — *Suppose $(M^\bullet, \phi)$ is a quasi-lift of $V^\bullet$ over some $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$. Then there exists a quasi-lift $(M'^\bullet, \phi')$ of $V^\bullet$ over R which is isomorphic to $(M^\bullet, \phi)$ and whose terms are topologically free pseudocompact R-modules.*

Proof. — Let $P^\bullet$ be a bounded above complex of topologically free pseudocompact $[[RG]]$-modules so that $f : P^\bullet \to M^\bullet$ is a quasi-isomorphism in $C^-([[RG]])$ which is surjective on terms. Then f induces an isomorphism

$$k \widehat{\otimes}_R^{\boldsymbol{L}} f : k \widehat{\otimes}_R P^\bullet = k \widehat{\otimes}_R^{\boldsymbol{L}} P^\bullet \longrightarrow k \widehat{\otimes}_R^{\boldsymbol{L}} M^\bullet$$

in $D^-([[kG]])$. Hence, by Remark 2.3, $M'^\bullet = P^\bullet$ together with $\phi' = \phi (k \widehat{\otimes}_R^{\boldsymbol{L}} f)$ has the required properties. $\square$

DEFINITION 2.10. — Let

$$\widehat{F} = \widehat{F}_{V^\bullet} : \widehat{\mathcal{C}} \longrightarrow \mathrm{Sets} \quad (\text{resp. } \widehat{F}^{\,\mathrm{fl}} = \widehat{F}_{V^\bullet}^{\,\mathrm{fl}} : \widehat{\mathcal{C}} \longrightarrow \mathrm{Sets})$$

be the map which sends an object R of $\widehat{\mathcal{C}}$ to the set $\widehat{F}(R)$ (resp. $\widehat{F}^{\,\mathrm{fl}}(R)$) of all deformations (resp. all proflat deformations) of $V^\bullet$ over R, and

which sends a morphism $\alpha: R \to R'$ in $\widehat{\mathcal{C}}$ to the set map $\widehat{F}(R) \to \widehat{F}(R')$ (resp. $\widehat{F}^{\,\mathrm{fl}}(R) \to \widehat{F}^{\,\mathrm{fl}}(R')$) induced by $M^\bullet \mapsto R' \widehat{\otimes}^{\boldsymbol{L}}_{R,\alpha} M^\bullet$. Let $F = F_{V^\bullet}$ (resp. $F^{\,\mathrm{fl}} = F^{\,\mathrm{fl}}_{V^\bullet}$) be the restriction of $\widehat{F}$ (resp. $\widehat{F}^{\,\mathrm{fl}}$) to the subcategory $\mathcal{C}$ of Artinian objects in $\widehat{\mathcal{C}}$. In the following, we will use the subscript $\mathcal{D}$ to denote the empty condition in case of the map $\widehat{F}$, and the condition of having topologically free cohomology groups in case of the map $\widehat{F}^{\,\mathrm{fl}}$. In particular, the notation $\widehat{F}_{\mathcal{D}}$ will be used to refer to both $\widehat{F}$ and $\widehat{F}^{\,\mathrm{fl}}$.

Let $k[\varepsilon]$, where $\varepsilon^2 = 0$, denote the ring of dual numbers over k. The set $F_{\mathcal{D}}(k[\varepsilon])$ is called the *tangent space* to $F_{\mathcal{D}}$, denoted by $t_{F_{\mathcal{D}}}$.

We first prove that $\widehat{F}_{\mathcal{D}}$ is a functor. We need the following result.

LEMMA 2.11. — *Suppose $(M^\bullet, \phi)$ is a proflat quasi-lift of $V^\bullet$ over some $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$. Then $\mathrm{H}^n(M^\bullet)$ is an abstractly free R-module of rank $d_n = \dim_k \mathrm{H}^n(V^\bullet)$ for all n. Moreover, for any $R' \in \mathrm{Ob}(\widehat{\mathcal{C}})$ and for any morphism $\alpha: R \to R'$ in $\widehat{\mathcal{C}}$, there is a natural R'-linear isomorphism $R' \widehat{\otimes}_R \mathrm{H}^n(M^\bullet) \cong \mathrm{H}^n(R' \widehat{\otimes}_R {}^{\boldsymbol{L}} M^\bullet)$.*

Proof. — By Lemma 2.9, we can assume that the terms of $M^\bullet$ are topologically free pseudocompact R-modules. Because $M^\bullet$ defines a proflat quasi-lift of $V^\bullet$ over R, the cohomology groups of $M^\bullet$ are topologically free pseudocompact R-modules. Since the complex $M^\bullet$ is bounded above, we obtain inductively that the terms of $M^\bullet$ split completely as

$$(2.1) \qquad M^n = \delta^{n-1}(C^{n-1}) \oplus Y^n \oplus C^n$$

where $\delta^{n-1}(C^{n-1}) = \mathrm{Image}(\delta^{n-1})$ and $\delta^{n-1}(C^{n-1}) \oplus Y^n = \mathrm{Ker}(\delta^n)$ as pseudocompact R-modules. Moreover, for any $R' \in \mathrm{Ob}(\widehat{\mathcal{C}})$ and any morphism $\alpha: R \to R'$ in $\widehat{\mathcal{C}}$, we have

$$R' \widehat{\otimes}_R M^n = (R' \widehat{\otimes}_R \delta^{n-1})(R' \widehat{\otimes}_R C^{n-1}) \oplus (R' \widehat{\otimes}_R Y^n) \oplus (R' \widehat{\otimes}_R C^n),$$

and inductively $\mathrm{H}^n(R' \widehat{\otimes}_R M^\bullet) = R' \widehat{\otimes}_R Y^n = R' \widehat{\otimes}_R \mathrm{H}^n(M^\bullet)$. Since this is in particular true for $R' = k$ and the natural surjection $\alpha: R \to k$, it follows that $\mathrm{H}^n(M^\bullet)$ is an abstractly free R-module of rank d_n for all n. This completes the proof of Lemma 2.11. $\qquad\square$

PROPOSITION 2.12. — *The map $\widehat{F}_{\mathcal{D}}$ is a functor $\widehat{\mathcal{C}} \to$ Sets. The functor $\widehat{F}^{\,\mathrm{fl}}$ is a subfunctor of $\widehat{F}$ in the sense that there is a natural transformation $\widehat{F}^{\,\mathrm{fl}} \to \widehat{F}$ which is injective. Moreover, if $V'^\bullet$ is a complex in $D^-([[kG]])$ satisfying Hypothesis 1 such that there is an isomorphism $\nu: V^\bullet \to V'^\bullet$ in $D^-([[kG]])$, then the natural transformation $\widehat{F}_{\mathcal{D}, V^\bullet} \to \widehat{F}_{\mathcal{D}, V'^\bullet}$ (resp. $F_{\mathcal{D}, V^\bullet} \to F_{\mathcal{D}, V'^\bullet}$) induced by $(M^\bullet, \phi) \mapsto (M^\bullet, \nu\phi)$ is an isomorphism of functors.*

Proof. — To show that $\widehat{F}_{\mathcal{D}}: \widehat{\mathcal{C}} \to$ Sets is a functor, it is enough to show the following. Suppose $R, R' \in \mathrm{Ob}(\widehat{\mathcal{C}})$, $\alpha: R \to R'$ is a morphism in $\widehat{\mathcal{C}}$ and $(M^\bullet, \phi) \in \widehat{F}_{\mathcal{D}}(R)$. Then $R' \widehat{\otimes}^{\boldsymbol{L}}_{R,\alpha} M^\bullet$ defines an element in $\widehat{F}_{\mathcal{D}}(R')$.

In case $\widehat{F}_{\mathcal{D}} = \widehat{F}$, this is obvious, since any pseudocompact R'-module S' is also a pseudocompact R-module via α. Hence

$$S' \widehat{\otimes}^{\boldsymbol{L}}_{R'}(R' \widehat{\otimes}^{\boldsymbol{L}}_{R,\alpha} M^\bullet) = S' \widehat{\otimes}^{\boldsymbol{L}}_R M^\bullet,$$

and, since $M^\bullet$ has finite pseudocompact R-tor dimension, it follows that $R' \widehat{\otimes}^{\boldsymbol{L}}_{R,\alpha} M^\bullet$ has finite pseudocompact R'-tor dimension. Thus $\widehat{F}$ is a functor. If $\widehat{F}_{\mathcal{D}} = \widehat{F}^{\,\mathrm{fl}}$, we additionally have to show that the cohomology groups of $R' \widehat{\otimes}^{\boldsymbol{L}}_{R,\alpha} M^\bullet$ are topologically free pseudocompact R'-modules. This follows from Lemma 2.11. Hence $\widehat{F}^{\,\mathrm{fl}}$ is a functor, and it is obvious that $\widehat{F}^{\,\mathrm{fl}}$ is a subfunctor of $\widehat{F}$. The last statement of Proposition 2.12 is also obvious. $\qquad\square$

DEFINITION 2.13. — A profinite group G has *finite pseudocompact cohomology*, if for each discrete $[[kG]]$-module M of finite k-dimension, and all integers j, the cohomology group $\mathrm{H}^j(G, M) = \mathrm{Ext}^j_{[[kG]]}(k, M)$ (as described in Remark 2.2 (iv)) has finite k-dimension.

We can now state our main result.

THEOREM 2.14. — *Suppose that G has finite pseudocompact cohomology.*

   (i) *The functor $F_{\mathcal{D}}$ has a pro-representable hull $R_{\mathcal{D}}(G, V^\bullet) \in \mathrm{Ob}(\widehat{\mathcal{C}})$ (cf. [25, Def. 2.7] and [20, §1.2]), and the functor $\widehat{F}_{\mathcal{D}}$ is continuous (cf. [20] and Definition 7.1).*

   (ii) *If $F_{\mathcal{D}} = F$, then there is a k-vector space isomorphism*

$$h: t_F \longrightarrow \mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet, V^\bullet).$$

If $F_{\mathcal{D}} = F^{\,\mathrm{fl}}$, then the composition of the natural map $t_{F^{\mathrm{fl}}} \to t_F$ and h induces an isomorphism between $t_{F^{\mathrm{fl}}}$ and the kernel of the natural map $\mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet, V^\bullet) \to \mathrm{Ext}^1_{D^-([[k]])}(V^\bullet, V^\bullet)$ given by forgetting the G-action.

   (iii) *If $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet) = k$, then $\widehat{F}_{\mathcal{D}}$ is represented by $R_{\mathcal{D}}(G, V^\bullet)$.*

Remark 2.15. — By Theorem 2.14 (i), there exists a deformation $U_{\mathcal{D}}(G, V^\bullet)$ of $V^\bullet$ over $R_{\mathcal{D}}(G, V^\bullet)$ with the following property.

For each $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$, the map $\mathrm{Hom}_{\widehat{\mathcal{C}}}(R_{\mathcal{D}}(G, V^\bullet), R) \to \widehat{F}_{\mathcal{D}}(R)$ induced by $\alpha \mapsto R \widehat{\otimes}^L_{R_{\mathcal{D}}(G,V^\bullet), \alpha} U_{\mathcal{D}}(G, V^\bullet)$ is surjective, and this map is bijective if R is the ring of dual numbers $k[\varepsilon]$ over k where $\varepsilon^2 = 0$.

In general, the isomorphism type of the pro-representable hull $R_{\mathcal{D}}(G, V^\bullet)$ is unique up to non-canonical isomorphism. If $R_{\mathcal{D}}(G, V^\bullet)$ represents $\widehat{F}_{\mathcal{D}}$, the pair $(R_{\mathcal{D}}(G, V^\bullet), U_{\mathcal{D}}(G, V^\bullet))$ is uniquely determined up to canonical isomorphism.

DEFINITION 2.16. — Using the notation of Theorem 2.14 and Remark 2.15, in case $\widehat{F}_{\mathcal{D}} = \widehat{F}$, we call

$R_{\mathcal{D}}(G, V^\bullet) = R(G, V^\bullet)$ the *versal deformation ring* of $V^\bullet$ and

$U_{\mathcal{D}}(G, V^\bullet) = U(G, V^\bullet)$ the *versal deformation* of $V^\bullet$.

In case $\widehat{F}_{\mathcal{D}} = \widehat{F}^{\,\mathrm{fl}}$, we call

$R_{\mathcal{D}}(G, V^\bullet) = R^{\,\mathrm{fl}}(G, V^\bullet)$ the *versal proflat deformation ring* of $V^\bullet$ and

$U_{\mathcal{D}}(G, V^\bullet) = U^{\,\mathrm{fl}}(G, V^\bullet)$ the *versal proflat deformation* of $V^\bullet$.

If $R_{\mathcal{D}}(G, V^\bullet)$ represents $\widehat{F}_{\mathcal{D}}$, then

$R(G, V^\bullet)$ (resp. $R^{\,\mathrm{fl}}(G, V^\bullet)$) will be called the *universal deformation ring* (resp. *the universal proflat deformation ring*) of $V^\bullet$, and

$U(G, V^\bullet)$ (resp. $U^{\,\mathrm{fl}}(G, V^\bullet)$) will be called the *universal deformation* (resp. *the universal proflat deformation*) of $V^\bullet$.

Remark 2.17. — (i) By part (ii) of Theorem 2.14, the tangent space $t_{F^{\mathrm{fl}}}$ consists of those elements

$$\gamma \in \mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet, V^\bullet) = \mathrm{Hom}_{D^-([[kG]])}(V^\bullet, T(V^\bullet))$$

which induce the trivial map on cohomology. In other words, the k-vector space maps $\gamma^i : \mathrm{H}^i(V^\bullet) \to \mathrm{H}^{i+1}(V^\bullet)$ which are induced by γ have to be zero for all i.

(ii) It follows from part (ii) of Theorem 2.14 that there exists a non-canonical surjective continuous W-algebra homomorphism $f_{\mathrm{fl}} : R(G, V^\bullet) \to R^{\,\mathrm{fl}}(G, V^\bullet)$.

(iii) If $V^\bullet$ consists of a single module V_0 in dimension 0, the versal deformation ring $R(G, V^\bullet)$ and the versal proflat deformation ring $R^{\,\mathrm{fl}}(G, V^\bullet)$ both coincide with the versal deformation ring studied by Mazur in [19], [20] (see Proposition 9.1). In this case, Mazur assumed only

that G satisfies a certain finiteness condition (Φ_p), which is equivalent to the requirement that $\mathrm{H}^1(G, M)$ have finite k-dimension for all discrete $[[kG]]$-modules M of finite k-dimension. Since the higher G-cohomology enters into determining lifts of complexes $V^\bullet$ having more than one non-zero cohomology group, the condition that G have finite pseudocompact cohomology is the natural generalization of Mazur's finiteness condition in this context.

(iv) Suppose k' is a finite extension of k, and W' is a complete local commutative Noetherian ring with residue field k' which is faithfully flat over W. In Theorem 8.1, we adapt an argument of Faltings from [28, Ch. 1] to show that $R_{\mathcal{D}}(G, k' \widehat{\otimes}^L_k V^\bullet) \cong W' \widehat{\otimes}_W R_{\mathcal{D}}(G, V^\bullet)$.

3. Computing deformations of complexes.

LEMMA 3.1. — *Suppose* $(M^\bullet, \phi)$ *is a quasi-lift of* $V^\bullet$ *over some Artinian object* $R \in \mathrm{Ob}(\mathcal{C})$. *Then* $\mathrm{H}^n(M^\bullet)$ *is a subquotient of an abstractly free* R-*module of rank* $d_n = \dim_k \mathrm{H}^n(V^\bullet)$ *for all* n. *In particular,* $M^\bullet$ *has only finitely many non-zero cohomology groups, all of which are discrete* R-*modules of finite length.*

Proof. — By Lemma 2.9, we can assume that the terms of $M^\bullet$ are topologically free pseudocompact R-modules. Thus, by Remark 2.4, the terms of $M^\bullet$ are abstractly free R-modules. Since we assume R to be Artinian, we can use the following fact about abstractly free R-modules. If $\{z_i\}_{i \in I}$ is a collection of elements of an abstractly free R-module N whose images in $k \widehat{\otimes}_R N$ form a k-basis, then $\{z_i\}_{i \in I}$ is an R-basis of N. In particular, if U is a subspace of $k \widehat{\otimes}_R N$, then we can extend any k-basis of U to a k-basis of $k \widehat{\otimes}_R N$. Hence we can lift a k-basis of U to a subset of an R-basis of N, and this subset is an R-basis of an abstractly free R-module N' which is a submodule and a direct summand of N such that $k \widehat{\otimes}_R N' = U$.

By assumption, the complex $k \widehat{\otimes}_R M^\bullet$ is isomorphic to $V^\bullet$ in $D^-([[kG]])$. In particular, the cohomology groups of $k \widehat{\otimes}_R M^\bullet$ have finite k-dimension, and almost all are zero. Consider the diagram

$$\cdots \longrightarrow M^{n-1} \xrightarrow{\;\delta^{n-1}\;} M^n \xrightarrow{\;\delta^n\;} M^{n+1} \longrightarrow \cdots$$
$$\left\downarrow{\scriptstyle \tau^{n-1}}\right. \qquad \left\downarrow{\scriptstyle \tau^n}\right. \qquad \left\downarrow{\scriptstyle \tau^{n+1}}\right.$$
$$\cdots \longrightarrow k \widehat{\otimes}_R M^{n-1} \xrightarrow{\;\delta^{n-1}_{(k)}\;} k \widehat{\otimes}_R M^n \xrightarrow{\;\delta^n_{(k)}\;} k \widehat{\otimes}_R M^{n+1} \longrightarrow \cdots$$

For all integers n, let r_n be the cardinality of a k-basis of $\mathrm{Image}(\delta^n_{(k)})$.

By lifting bases, we find an abstractly free R-module C^{n-1} of rank r_{n-1} which is a submodule and a direct summand of the abstractly free R-module M^{n-1} such that $\tau^n(\delta^{n-1}(C^{n-1})) = \text{Image}(\delta^{n-1}_{(k)})$. We can also lift bases to obtain an abstractly free R-module Y^n of rank d_n which is a submodule and a direct summand of M^n such that

$$k \widehat{\otimes}_R M^n = \tau^n\big(\delta^{n-1}(C^{n-1})\big) \oplus \tau^n(Y^n) \oplus \tau^n(C^n).$$

Hence by the remark at the beginning of the proof,

$$M^n = \delta^{n-1}(C^{n-1}) \oplus Y^n \oplus C^n.$$

Since C^n and $\delta^n(C^n)$ are both abstractly free R-modules of rank r_n, it follows that $\text{Ker}(\delta^n)$ is contained in the abstractly free R-module $\delta^{n-1}(C^{n-1}) \oplus Y^n$. On the other hand, the abstractly free R-module $\delta^{n-1}(C^{n-1})$ is contained in $\text{Image}(\delta^{n-1})$. Hence $\text{H}^n(M^\bullet)$ is a subquotient of the abstractly free R-module Y^n of rank d_n. $\qquad\square$

DEFINITION 3.2. — Let $R \in \text{Ob}(\mathcal{C})$ be Artinian. Define $D^-_{\text{fin}}([[RG]])$ (resp. $K^-_{\text{fin}}([[RG]])$, resp. $C^-_{\text{fin}}([[RG]])$) to be the full subcategory of $D^-([[RG]])$ (resp. $K^-([[RG]])$, resp. $C^-([[RG]])$) whose objects are those complexes $M^\bullet$ of finite pseudocompact R-tor dimension having finitely many non-zero cohomology groups, all of which have finite R-length. Suppose Δ and Δ' are closed normal subgroups of G, and $\Delta' \subseteq \Delta$. Inflation from G/Δ to G/Δ' defines a functor

$$\text{Inf}^{G/\Delta'}_{G/\Delta} : D^-_{\text{fin}}\big([R(G/\Delta)]\big) \longrightarrow D^-_{\text{fin}}\big([R(G/\Delta')]\big).$$

Remark 3.3. — Suppose R is an Artinian object in $\text{Ob}(\mathcal{C})$. By Remark 2.2 (iii), an $[[RG]]$-module has finite length if and only if it has finite length as R-module. Since R is local Artinian, an R-module has finite R-length if and only if it has finite k-length.

LEMMA 3.4. — Suppose $R \in \text{Ob}(\mathcal{C})$ is Artinian, and $N^\bullet$ is an object in $C^-_{\text{fin}}([[RG]])$.

(i) Suppose $\text{H}^j(N^\bullet) = 0$ for $j < n$. Then there is an exact sequence of complexes

(3.1) $$0 \to U^\bullet \xrightarrow{\iota} N^\bullet \longrightarrow N'^\bullet \to 0$$

in $C^-_{\text{fin}}([[RG]])$ such that $U^\bullet$ is acyclic, and such that the terms of $N'^\bullet$ have finite k-length and satisfy $N'^j = 0$ for $j < n$.

(ii) Suppose $N_1^\bullet$ and $N_2^\bullet$ are two complexes in $C^-_{\text{fin}}([[RG]])$, such that all terms of $N_1^\bullet$ (resp. $N_2^\bullet$) have finite k-length and satisfy $N_1^j = 0 = N_2^j$ for $j < n$. Suppose there exist morphisms $N_1^\bullet \xleftarrow{g_1} N^\bullet \xrightarrow{g_2} N_2^\bullet$ in $C^-_{\text{fin}}([[RG]])$ such that g_1 is a quasi-isomorphism. Then there is an exact sequence of complexes (3.1) in $C^-_{\text{fin}}([[RG]])$ which satisfies the properties in part (i) and additionally $g_1 \circ \iota = 0 = g_2 \circ \iota$.

Proof. — For part (i), suppose $\text{H}^j(N^\bullet) = 0$ for $j < n$. Define $U_0^\bullet$ to be the complex which is equal to $N^\bullet$ in dimensions less than n, $U_0^n = \delta^{n-1}(N^{n-1})$, and $U_0^j = 0$ if $j > n$. Then $U_0^\bullet$ is an acyclic subcomplex of $N^\bullet$. By dividing $N^\bullet$ by $U_0^\bullet$, we can assume $N^j = 0$ for $j < n$. Hence $N^\bullet$ is now a bounded complex, since it is bounded above. If all the terms of $N^\bullet$ have finite R-length, we let $U^\bullet$ be the zero complex.

Suppose now that there is a smallest integer $m \geq n$ such that N^m has infinite R-length, but N^{m-1} has finite R-length. Let $Z^m \subseteq N^m$ be the kernel of the boundary map $\delta^m : N^m \to N^{m+1}$. Because N^{m-1} has finite R-length, and $\text{H}^m(N^\bullet) = Z^m/\delta^{m-1}(N^{m-1})$ has finite R-length by assumption, Z^m has finite R-length. The pseudocompact R-module N^m admits a basis $\mathcal{U}$ of open neighborhoods of 0 consisting of submodules U for which N^m/U has finite R-length. Therefore, $\bigcap_{U \in \mathcal{U}} U = \{0\}$, and hence $\bigcap_{U \in \mathcal{U}}(Z^m \cap U) = \{0\}$. Since Z^m has finite R-length, and $\mathcal{U}$ is a basis of neighborhoods of 0, we conclude that there is an element $U \in \mathcal{U}$ with $Z^m \cap U = \{0\}$. Note now that $\widetilde{N}^m = N^m/U$ has finite R-length, and the boundary map $\delta^m : N^m \to N^{m+1}$ defines an isomorphism between U and $\delta^m(U)$ because the intersection of U with $Z^m = \text{Ker}(\delta^m)$ is trivial.

We now let $U_1^\bullet$ be the acyclic complex which has term $\{0\}$ in dimensions other than m and $m+1$ and which in dimensions m and $m+1$ is given by the isomorphism $U \to \delta^m(U)$. Then $U_1^\bullet$ is naturally an acyclic subcomplex of $N^\bullet$, so we can form the quotient complex $\widetilde{N}^\bullet = N^\bullet/U_1^\bullet$. Since $\widetilde{N}^m = N^m/U$ has finite R-length, the number of terms of $\widetilde{N}^\bullet$ which have infinite R-length is one less than the number of such terms of $N^\bullet$. Continuing this way, we arrive at an exact sequence (3.1) of complexes as described in part (i) of Lemma 3.4.

Part (ii) is proved by adjusting the arguments in the proof of part (i) to the situation of part (ii). Let $U_0^\bullet$ be as above, i.e. $U_0^\bullet$ is the complex which is equal to $N^\bullet$ in dimensions less than n, $U_0^n = \delta^{n-1}(N^{n-1})$, and $U_0^j = 0$ for $j > n$. Since $N_1^j = 0 = N_2^j$ for $j < n$ and g_1 is a quasi-isomorphism, $U_0^\bullet$ is an acyclic subcomplex of $N^\bullet$, and $g_{1|U_0^\bullet} = 0 = g_{2|U_0^\bullet}$. Hence, by dividing $N^\bullet$ by $U_0^\bullet$, we can assume $N^j = 0$ for $j < n$. If all terms

of $N^\bullet$ have finite k-length, we let $U^\bullet$ be the zero complex. Suppose now that there exists a smallest integer $m \geq n$ such that N^m has infinite k-length, but N^{m-1} has finite k-length. Let $Z^m = \mathrm{Ker}(\delta^m)$ and let $\mathcal{U}$ be a basis of open neighborhoods of 0 for N^m as above. Then it follows, as in part (i), that there exists an element $U \in \mathcal{U}$ with $Z^m \cap U = \{0\}$. Since N_1^m (resp. N_2^m) has finite k-length, it follows that there exists an element $U_1 \in \mathcal{U}$ (resp. $U_2 \in \mathcal{U}$) such that $g_1^m{}_{|U_1} = 0$ (resp. $g_2^m{}_{|U_2} = 0$). Since $\mathcal{U}$ is a basis of open neighborhoods of 0, there exists an element $U' \in \mathcal{U}$ contained in $U \cap U_1 \cap U_2$. Hence, after replacing U by U', we obtain $Z^m \cap U = \{0\}$, and $g_1^m{}_{|U} = 0 = g_2^m{}_{|U}$. We now define the complex $U_1^\bullet$ as above, using this element U. It follows that $U_1^\bullet$ is an acyclic subcomplex of $N^\bullet$, and $g_{1|U_1^\bullet} = 0 = g_{2|U_1^\bullet}$. Since the number of terms of $N^\bullet/U_1^\bullet$ which have infinite k-length is one less than the number of such terms of $N^\bullet$, we can use induction to obtain an exact sequence (3.1) of complexes having the desired properties described in part (ii) of Lemma 3.4. $\square$

Remark 3.5. — Suppose $R \in \mathrm{Ob}(\mathcal{C})$ is Artinian and Δ_0 is a closed normal subgroup of finite index in G. Let $N_1^\bullet, N_2^\bullet$ be complexes in $D_{\mathrm{fin}}^-([R(G/\Delta_0)])$ such that all their terms have finite k-length, and let $g : N_1^\bullet \to N_2^\bullet$ be a morphism in $D_{\mathrm{fin}}^-([R(G/\Delta_0)])$. By Remark 2.2 (ii), and since $[R(G/\Delta_0)]$ is Noetherian, there exist bounded above complexes $M_1^\bullet$ and $M_2^\bullet$ of abstractly free finitely generated $[R(G/\Delta_0)]$-modules such that there is an isomorphism $\beta_i : N_i^\bullet \to M_i^\bullet$ in $D_{\mathrm{fin}}^-([R(G/\Delta_0)])$ $(i = 1, 2)$. Then $f = \beta_2\, g\, \beta_1^{-1}$ is a morphism $f : M_1^\bullet \to M_2^\bullet$ in $D_{\mathrm{fin}}^-([R(G/\Delta_0)])$. Let $\mathcal{P}$ be the the additive subcategory of $\mathrm{PCMod}([R(G/\Delta_0)])$ of projective objects. By the dual of [17, Prop. I.4.7], the natural functor $K^-(\mathcal{P}) \to D^-([R(G/\Delta_0)])$ is an equivalence of categories. Hence f can be taken to be a morphism in $K_{\mathrm{fin}}^-([R(G/\Delta_0)])$.

COROLLARY 3.6. — *Suppose $R \in \mathrm{Ob}(\mathcal{C})$ is Artinian, and $N^\bullet$, $N_1^\bullet$ and $N_2^\bullet$ are objects in $D_{\mathrm{fin}}^-([[RG]])$. Let $g : N_1^\bullet \to N_2^\bullet$ be a morphism in $D_{\mathrm{fin}}^-([[RG]])$.*

(i) *There exists a closed normal subgroup Δ of finite index in G with the following property: There is a bounded above complex $M^\bullet$ of abstractly free finitely generated $[R(G/\Delta)]$-modules, and an isomorphism $\beta : N^\bullet \to \mathrm{Inf}_{G/\Delta}^G(M^\bullet)$ in $D_{\mathrm{fin}}^-([[RG]])$.*

(ii) *There exists a closed normal subgroup Δ of finite index in G with the following property: There are bounded above complexes $M_1^\bullet$ and $M_2^\bullet$ of abstractly free finitely generated $[R(G/\Delta)]$-modules, a morphism*

$f : M_1^\bullet \to M_2^\bullet$ *in* $K_{\mathrm{fin}}^-([R(G/\Delta)])$, *and isomorphisms* $\beta_i : N_i^\bullet \to \mathrm{Inf}_{G/\Delta}^G(M_i^\bullet)$ *in* $D_{\mathrm{fin}}^-([[RG]])$ $(i = 1, 2)$ *such that* $\mathrm{Inf}_{G/\Delta}^G(f) = \beta_2\, g\, \beta_1^{-1}$ *as morphisms in* $D_{\mathrm{fin}}^-([[RG]])$.

Moreover, suppose Δ is as in part (i) (*resp. part* (ii)), *and Δ' is a closed normal subgroup of finite index in G with $\Delta' \subset \Delta$. Then Δ' has the same property as Δ has in part* (i) (*resp. part* (ii)) *with Δ replaced by Δ'.*

Proof. — By Lemma 3.4 (i), there exists a bounded complex $N'^\bullet$ such that the terms of $N'^\bullet$ have finite k-length and such that $N^\bullet$ is isomorphic to $N'^\bullet$ in $D_{\mathrm{fin}}^-([[RG]])$. Since all the terms of $N'^\bullet$ have finite k-length, there exists an open (and closed) normal subgroup Δ of finite index in G which acts trivially on all the terms of $N'^\bullet$. If Δ' is closed normal of finite index in G with $\Delta' \subset \Delta$, then Δ' also acts trivially on all the terms of $N'^\bullet$. Let now $\Delta_0 = \Delta$ or Δ'. Then $N'^\bullet$ can be viewed as a complex in $D_{\mathrm{fin}}^-([R(G/\Delta_0)])$. Part (i) follows now from Remark 3.5.

We now prove part (ii). By Lemma 3.4 (i), we may assume that $N_1^\bullet$ and $N_2^\bullet$ are bounded such that all their terms have finite k-length. The morphism $g : N_1^\bullet \to N_2^\bullet$ in $D_{\mathrm{fin}}^-([[RG]])$ is represented by a pair of morphisms in $C_{\mathrm{fin}}^-([[RG]])$ of the form

$$\text{(3.2)} \qquad \begin{array}{ccc} & T^\bullet & \\ {\scriptstyle g_1}\swarrow & & \searrow{\scriptstyle g_2} \\ N_1^\bullet & & N_2^\bullet \end{array}$$

where g_1 is a quasi-isomorphism. It follows from Lemma 3.4 (ii) that we can divide $T^\bullet$ by an acyclic complex so as to be able to assume that $T^\bullet$ in (3.2) is also bounded and its terms have finite k-length. Hence there exists an open (and closed) normal subgroup Δ of finite index in G which acts trivially on all the terms of $N_1^\bullet$, $N_2^\bullet$ and $T^\bullet$. If Δ' is closed normal of finite index in G with $\Delta' \subset \Delta$, then Δ' also acts trivially on all the terms of $N_1^\bullet$, $N_2^\bullet$ and $T^\bullet$. Let now $\Delta_0 = \Delta$ or Δ'. Then we can view $N_1^\bullet$, $N_2^\bullet$ and $T^\bullet$ as complexes in $D_{\mathrm{fin}}^-([R(G/\Delta_0)])$. Thus (3.2) defines a morphism $N_1^\bullet \to N_2^\bullet$ in the derived category $D_{\mathrm{fin}}^-([R(G/\Delta_0)])$. Part (ii) follows now from Remark 3.5. This completes the proof of Corollary 3.6. $\square$

DEFINITION 3.7. — In the situation of Corollary 3.6 (i), we say *we can replace $N^\bullet$ by $M^\bullet$*. In the situation of Corollary 3.6 (ii), we say *we can replace $N_i^\bullet$ by $M_i^\bullet$* $(i = 1, 2)$, and *g by f*.

LEMMA 3.8. — *Suppose $M^\bullet$ is an object in $D_{\mathrm{fin}}^-([[RG]])$ such that $\mathrm{H}^j(M^\bullet) = 0$ for $j < n$. Then $M^\bullet$ has finite pseudocompact R-tor dimension at n.*

Proof. — By Corollary 3.6 (i), we may assume that $M^\bullet$ is a bounded above complex of abstractly free finitely generated $[R(G/\Delta)]$-modules for some closed normal subgroup Δ of finite index in G. Hence all terms of $M^\bullet$ are abstractly free finitely generated R-modules. By Remark 2.6, there exists an integer $n_1 \leq n$ such that $M^{n_1}/\delta^{n_1-1}(M^{n_1-1})$ is a topologically free pseudocompact R-module. Since R is Artinian, it follows by Remark 2.4 that this is an abstractly free R-module. To prove Lemma 3.8, it is enough to show that $M^n/\delta^{n-1}(M^{n-1})$ is an abstractly free R-module. If $n_1 = n$, there is nothing to show. Suppose now that $n_1 < n$. Since $\mathrm{H}^{n_1}(M^\bullet) = 0$, it follows that $\delta^{n_1}(M^{n_1}) \cong M^{n_1}/\delta^{n_1-1}(M^{n_1-1})$, and thus $\delta^{n_1}(M^{n_1})$ is an abstractly free R-module. Inclusion gives an injective R-module homomorphism

$$\iota^{n_1} : \delta^{n_1}(M^{n_1}) \longrightarrow M^{n_1+1},$$

and we claim that the reduction modulo the maximal ideal m_R of R

$$\iota^{n_1}_{(k)} : \delta^{n_1}_{(k)}(k \,\widehat{\otimes}_R M^{n_1}) \longrightarrow k \,\widehat{\otimes}_R M^{n_1+1}$$

stays injective. Otherwise, since $\delta^{n_1}(M^{n_1})$ and M^{n_1+1} are abstractly free R-modules, there exists an element $x \in \delta^{n_1}(M^{n_1})$, $x \notin m_R \cdot \delta^{n_1}(M^{n_1})$, with $\iota^{n_1}(x) \in m_R \cdot M^{n_1+1}$. Since R is Artinian, there exists a non-zero element $t \in R$ such that t annihilates m_R and $t\,x \neq 0$. Then $\iota^{n_1}(t\,x) = t\,\iota^{n_1}(x) = 0$, which contradicts the injectivity of ι^{n_1}. Thus $\iota^{n_1}_{(k)}$ is injective, and it follows from [21, Lemma IV.1.11] that ι^{n_1} has a section which is an R-module homomorphism. Hence the short exact sequence

$$0 \to \delta^{n_1}(M^{n_1}) \xrightarrow{\iota^{n_1}} M^{n_1+1} \longrightarrow M^{n_1+1}/\delta^{n_1}(M^{n_1}) \to 0$$

splits, and $M^{n_1+1}/\delta^{n_1}(M^{n_1})$ is an abstractly free R-module. Inductively, using that $\mathrm{H}^j(M^\bullet) = 0$ for $j < n$, it follows that $M^n/\delta^{n-1}M^{n-1}$ is an abstractly free R-module. This proves Lemma 3.8. □

LEMMA 3.9. — *Suppose $R, R_1, R_2 \in \mathrm{Ob}(\mathcal{C})$ are Artinian with morphisms $R_1 \xrightarrow{\alpha_1} R \xleftarrow{\alpha_2} R_2$ in $\mathcal{C}$. Let $N^\bullet$ be an object in $C^-_{\mathrm{fin}}([[RG]])$. Suppose Δ_1, Δ_2 are closed normal subgroups of finite index in G, and let $X^\bullet_i$ be a complex in $C^-_{\mathrm{fin}}([(R_i(G/\Delta_i)])$ of abstractly free finitely generated $[R_i(G/\Delta_i)]$-modules such that $\mathrm{H}^j(X^\bullet_i) = 0$ for $j < n$ $(i = 1,2)$. Suppose there exist morphisms*

$$(3.3) \qquad R \,\widehat{\otimes}_{R_1} \mathrm{Inf}^G_{G/\Delta_1} X^\bullet_1 \xleftarrow{\tau_1} N^\bullet \xrightarrow{\tau_2} R \widehat{\otimes}_{R_2} \mathrm{Inf}^G_{G/\Delta_2} X^\bullet_2$$

in $C^-_{\mathrm{fin}}([[RG]])$ such that τ_1 is a quasi-isomorphism. Then there is an exact sequence of complexes (3.1) $0 \to U^\bullet \xrightarrow{\iota} N^\bullet \to N'^\bullet \to 0$ in $C^-_{\mathrm{fin}}([[RG]])$ which satisfies the properties of Lemma 3.4 (i). Moreover, there is an exact sequence of complexes

$$(3.4) \qquad\qquad 0 \to E^\bullet_i \longrightarrow X^\bullet_i \xrightarrow{\pi_i} X'^\bullet_i \to 0$$

in $C^-_{\mathrm{fin}}([R_i(G/\Delta_i)])$ with the following properties: The complex $E^\bullet_i$ is acyclic, the terms of $X'^\bullet_i$ are abstractly free R-modules of finite k-length and $X'^j_i = 0$ for $j < n$, such that $R\,\widehat{\otimes}_{R_i}\pi_i$ is a quasi-isomorphism in $C^-_{\mathrm{fin}}([R(G/\Delta)])$ and $(R\,\widehat{\otimes}_{R_i} \mathrm{Inf}^G_{G/\Delta_i} \pi_i) \circ \tau_i \circ \iota = 0$ $(i = 1,2)$.

Proof. — For $i = 1, 2$, let $E^\bullet_i$ be the acyclic subcomplex of $X^\bullet_i$ which is equal to $X^\bullet_i$ in dimensions less than n, $E^n_i = \delta^{n-1}(X^{n-1}_i)$, and $E^j_i = 0$ for $j > n$. Let $X'^\bullet_i = X^\bullet_i/E^\bullet_i$ and let $\pi_i : X^\bullet_i \to X'^\bullet_i$ be the natural morphism in $C^-_{\mathrm{fin}}([R_i(G/\Delta_i)])$. Define $\tilde{\pi}_i = \mathrm{Inf}^G_{G/\Delta_i} \pi_i$. Since G/Δ_i is a finite group, all terms of $X'^\bullet_i$ have finite k-length. By Lemma 3.8 and Remarks 2.6 and 2.4, the terms of $X'^\bullet_i$ are abstractly free R-modules. Hence $R\,\widehat{\otimes}_{R_i} X'^\bullet_i = R\,\widehat{\otimes}^L_{R_i} X'^\bullet_i$, and $R\,\widehat{\otimes}_{R_i}\tilde{\pi}_i = R\,\widehat{\otimes}^L_{R_i}\tilde{\pi}_i$ is a quasi-isomorphism in $C^-_{\mathrm{fin}}([[RG]])$. Thus we have morphisms

$$R \,\widehat{\otimes}_{R_1} \mathrm{Inf}^G_{G/\Delta_1} X'^\bullet_1 \xleftarrow{(R\,\widehat{\otimes}_{R_1}\tilde{\pi}_1)\circ\tau_1} N^\bullet \xrightarrow{(R\,\widehat{\otimes}_{R_2}\tilde{\pi}_2)\circ\tau_2} R\,\widehat{\otimes}_{R_2} \mathrm{Inf}^G_{G/\Delta_2} X'^\bullet_2$$

in $C^-_{\mathrm{fin}}([[RG]])$ and $(R\,\widehat{\otimes}_{R_1}\tilde{\pi}_1) \circ \tau_1$ is a quasi-isomorphism. Lemma 3.9 follows now from Lemma 3.4 (ii). □

The next two corollaries will be used in §5 to verify Schlessinger's axioms (H1), (H2), (H4), and in §7 to prove that $\widehat{F}_{\mathcal{D}}$ is continuous.

COROLLARY 3.10. — *Suppose $(R_i)^r_{i=1}, (S_i)^r_{i=1}, (T_{ij})_{1\leq i \leq r, 1 \leq j \leq s}$ are finite collections of Artinian objects in $\mathrm{Ob}(\mathcal{C})$ with morphisms*

$$R_i \xrightarrow{\alpha_{ij}} T_{ij} \xleftarrow{\beta_{ij}} S_i$$

in $\mathcal{C}$ for $1 \leq i \leq r$, $1 \leq j \leq s$. Let $X^\bullet_i$ be an object in $D^-_{\mathrm{fin}}([[R_iG]])$, and let $Z^\bullet_i$ be an object in $D^-_{\mathrm{fin}}([[S_iG]])$. Let $\tau_{ij} : T_{ij} \widehat{\otimes}^L_{S_i} Z^\bullet_i \to T_{ij} \widehat{\otimes}^L_{R_i} X^\bullet_i$ be a morphism in $D^-_{\mathrm{fin}}([[T_{ij}G]])$, and let $\xi_i : k\,\widehat{\otimes}^L_{R_i} X^\bullet_i \to V^\bullet$ (resp. $\zeta_i : k\,\widehat{\otimes}^L_{S_i} Z^\bullet_i \to V^\bullet$) be a morphism in $D^-_{\mathrm{fin}}([[kG]])$, with $\zeta_i = \xi_i \,(k\,\widehat{\otimes}^L_{T_{ij}}\tau_{ij})$ $(i = 1, \ldots, r, j = 1, \ldots, s)$.

Then there exists a closed normal subgroup Δ of finite index in G with the following property: We can replace $V^\bullet$ by a bounded above complex $\widetilde{V}^\bullet$

of abstractly free finitely generated $[k(G/\Delta)]$-modules, and we can replace $X_i^\bullet$ (resp. $Z_i^\bullet$) by a bounded above complex $\widetilde{X}_i^\bullet$ (resp. $\widetilde{Z}_i^\bullet$) of abstractly free finitely generated $[R_i(G/\Delta)]$-modules (resp. $[S_i(G/\Delta)]$-modules). We can replace τ_{ij} by $\tilde{\tau}_{ij}\colon T_{ij}\otimes_{S_i}\widetilde{Z}_i^\bullet \to T_{ij}\otimes_{R_i}\widetilde{X}_i^\bullet$ in $K_{\mathrm{fin}}^-([T_{ij}(G/\Delta)])$, and we can replace ξ_i (resp. ζ_i) by $\tilde{\xi}_i\colon k\otimes_{R_i}\widetilde{X}_i^\bullet \to \widetilde{V}^\bullet$ (resp. $\tilde{\zeta}_i\colon k\otimes_{S_i}\widetilde{Z}_i^\bullet \to \widetilde{V}^\bullet$) in $K_{\mathrm{fin}}^-([k(G/\Delta)])$, such that $\tilde{\zeta}_i = \tilde{\xi}_i\,(k\otimes_{T_{ij}}\tilde{\tau}_{ij})$ $(i=1,\ldots,r,\; j=1,\ldots,s)$.

Moreover, suppose Δ' is a closed normal subgroup of finite index in G with $\Delta' \subset \Delta$. Then Δ' has the same property as Δ has with Δ replaced by Δ'.

Proof. — By Corollary 3.6 (i), there exists a closed normal subgroup Δ_1 of finite index in G such that we can replace $V^\bullet$ by a bounded above complex of abstractly free finitely generated $[k(G/\Delta_1)]$-modules and such that we can replace $X_i^\bullet$ (resp. $Z_i^\bullet$) by a bounded above complex of abstractly free finitely generated $[R_i(G/\Delta_1)]$-modules (resp. $[S_i(G/\Delta_1)]$-modules) $(i=1,\ldots,r)$. Then, for $i = 1,\ldots,r$, $j = 1,\ldots,s$, τ_{ij} is a morphism $\mathrm{Inf}_{G/\Delta_1}^G(T_{ij}\otimes_{S_i}Z_i^\bullet) \to \mathrm{Inf}_{G/\Delta_1}^G(T_{ij}\otimes_{R_i}X_i^\bullet)$ in $D_{\mathrm{fin}}^-([[T_{ij}G]])$ and is represented by a pair of morphisms in $C_{\mathrm{fin}}^-([[T_{ij}G]])$ of the form

$$(3.5)\qquad
\begin{array}{ccc}
& Y_{ij}^\bullet & \\
{}^{\tau_{ij1}}\swarrow & & \searrow^{\tau_{ij2}} \\
\mathrm{Inf}_{G/\Delta_1}^G(T_{ij}\otimes_{S_i}Z_i^\bullet) & & \mathrm{Inf}_{G/\Delta_1}^G(T_{ij}\otimes_{R_i}X_i^\bullet)
\end{array}$$

where τ_{ij1} is a quasi-isomorphism. Similarly, ξ_i (resp. ζ_i) is a morphism $\mathrm{Inf}_{G/\Delta_1}^G(k\otimes_{R_i}X_i^\bullet) \to \mathrm{Inf}_{G/\Delta_1}^G V^\bullet$ (resp. $\mathrm{Inf}_{G/\Delta_1}^G(k\otimes_{S_i}Z_i^\bullet) \to \mathrm{Inf}_{G/\Delta_1}^G V^\bullet$) in $D_{\mathrm{fin}}^-([[kG]])$ and is represented by a pair of morphisms in $C_{\mathrm{fin}}^-([[kG]])$ of the form

$$(3.6)\qquad
\begin{array}{ccc}
& A_i^\bullet & \\
{}^{\xi_{i1}}\swarrow & & \searrow^{\xi_{i2}} \\
\mathrm{Inf}_{G/\Delta_1}^G(k\otimes_{R_i}X_i^\bullet) & & \mathrm{Inf}_{G/\Delta_1}^G V^\bullet
\end{array}$$

(resp.

$$
\begin{array}{ccc}
& B_i^\bullet & \\
{}^{\zeta_{i1}}\swarrow & & \searrow^{\zeta_{i2}} \\
\mathrm{Inf}_{G/\Delta_1}^G(k\otimes_{S_i}Z_i^\bullet) & & \mathrm{Inf}_{G/\Delta_1}^G V^\bullet\,),
\end{array}$$

where ξ_{i1} (resp. ζ_{i1}) is a quasi-isomorphism. There exists an integer n such that for $t < n$, $\mathrm{H}^t(V^\bullet) = 0$, $\mathrm{H}^t(X_i^\bullet) = 0$, and $\mathrm{H}^t(Z_i^\bullet) = 0$ $(i=1,\ldots,r)$. Let now $\mathcal{S}$ be the set consisting of all the complexes $V^\bullet$, $X_i^\bullet$, $Z_i^\bullet$, $Y_{ij}^\bullet$, $A_i^\bullet$, $B_i^\bullet$ $(i=1,\ldots,r,\; j=1,\ldots,s)$, and let $\Sigma^\bullet \in \mathcal{S}$. It follows from Lemma 3.9 that

we can divide $\Sigma^\bullet$ by a suitable acyclic subcomplex to be able to assume that $\Sigma^\bullet$ in (3.5) (resp. (3.6)) is bounded and its terms have finite k-length. Thus there exists an open (and closed) normal subgroup Δ of finite index in G with $\Delta \subseteq \Delta_1$ such that Δ acts trivially on all the terms of $\Sigma^\bullet$ for all $\Sigma^\bullet \in \mathcal{S}$. If Δ' is closed normal of finite index in G with $\Delta' \subset \Delta$, then Δ' also acts trivially on all the terms of $\Sigma^\bullet$ for all $\Sigma^\bullet \in \mathcal{S}$. Let now $\Delta_0 = \Delta$ or Δ'. Then we can view $\Sigma^\bullet$ as a complex in $D_{\mathrm{fin}}^-([\Omega_\Sigma(G/\Delta_0)])$, where Ω_Σ is the coefficient ring belonging to Σ. Thus (3.5) (resp. (3.6)) define morphisms in $D_{\mathrm{fin}}^-([T_{ij}(G/\Delta_0)])$ (resp. in $D_{\mathrm{fin}}^-([k(G/\Delta_0)])$). By Remark 3.5, we obtain the required complexes $\widetilde{V}^\bullet$, $\widetilde{X}_i^\bullet$ and $\widetilde{Z}_i^\bullet$, and the required morphisms $\tilde{\tau}_{ij}$ in $K_{\mathrm{fin}}^-([T_{ij}(G/\Delta_0)])$ (resp. $\tilde{\xi}_i$ and $\tilde{\zeta}_i$ in $K_{\mathrm{fin}}^-([k(G/\Delta_0)])$) $(i=1,\ldots,r,\; j=1,\ldots,s)$. It is obvious that the equality $\tilde{\zeta}_i = \tilde{\xi}_i\,(k\otimes_{T_{ij}}\tilde{\tau}_{ij})$ is preserved for all i,j. $\qquad\square$

Corollary 3.11. — *Suppose $R_1, R_2 \in \mathrm{Ob}(\mathcal{C})$ are Artinian with a surjective morphism $\alpha\colon R_2 \to R_1$ in $\mathcal{C}$. Suppose Δ is a closed normal subgroup of finite index in G, and suppose $V^\bullet$ is a complex in $D_{\mathrm{fin}}^-([k(G/\Delta)])$ of abstractly free finitely generated $[k(G/\Delta)]$-modules. Suppose $M_i^\bullet$ is a complex in $D_{\mathrm{fin}}^-([[R_iG]])$ such that there is a morphism $\phi_i\colon k\widehat{\otimes}_{R_i}^{\boldsymbol{L}}M_i^\bullet \to \mathrm{Inf}_{G/\Delta}^G V^\bullet$ in $D_{\mathrm{fin}}^-([[kG]])$ $(i=1,2)$.*

(i) For $i = 1,2$, there exists a closed normal subgroup Δ_i of finite index in G with $\Delta_i \subseteq \Delta$ having the following property: We can replace $M_i^\bullet$ by a bounded above complex $N_i^\bullet$ of abstractly free finitely generated $[R_i(G/\Delta_i)]$-modules, and we can replace ϕ_i by $\psi_i\colon k\widehat{\otimes}_{R_i}N_i^\bullet \to \mathrm{Inf}_{G/\Delta}^{G/\Delta_i}V^\bullet$ in $K_{\mathrm{fin}}^-([k(G/\Delta_i)])$.

(ii) Suppose $\rho\colon R_1\widehat{\otimes}_{R_2}\mathrm{Inf}_{G/\Delta_2}^G N_2^\bullet \to \mathrm{Inf}_{G/\Delta_1}^G N_1^\bullet$ is a morphism in $D_{\mathrm{fin}}^-([[R_1G]])$ satisfying $\mathrm{Inf}_{G/\Delta_1}^G(\psi_1)\,(k\widehat{\otimes}_{R_1}\rho) = \mathrm{Inf}_{G/\Delta_2}^G(\psi_2)$. Then there exists a closed normal subgroup $\widetilde{\Delta}_2$ of finite index in G with $\widetilde{\Delta}_2 \subseteq \Delta_1 \cap \Delta_2$ having the following property: We can replace $N_2^\bullet$ by a complex $\widetilde{N}_2^\bullet$ of abstractly free finitely generated $[R_2(G/\widetilde{\Delta}_2)]$-modules, and we can replace ψ_2 by $\widetilde{\psi}_2\colon k\widehat{\otimes}_{R_2}\widetilde{N}_2^\bullet \to \mathrm{Inf}_{G/\Delta}^{G/\widetilde{\Delta}_2}V^\bullet$ in $K_{\mathrm{fin}}^-([k(G/\widetilde{\Delta}_2)])$. We can replace ρ by $\tilde{\rho}\colon R_1\widehat{\otimes}_{R_2}\widetilde{N}_2^\bullet \to \mathrm{Inf}_{G/\Delta_1}^{G/\widetilde{\Delta}_2}N_1^\bullet$ in $K_{\mathrm{fin}}^-([R_1(G/\widetilde{\Delta}_2)])$ such that $\mathrm{Inf}_{G/\Delta_1}^{G/\widetilde{\Delta}_2}(\psi_1)\,(k\widehat{\otimes}_{R_1}\tilde{\rho}) = \widetilde{\psi}_2$.

Proof. — We first prove part (i). Let $i \in \{1, 2\}$. By Corollary 3.6 (i), there exists a closed normal subgroup Δ_i' of finite index in G with $\Delta_i' \subseteq \Delta$ such that we can replace $M_i^\bullet$ by a bounded above complex of abstractly

free finitely generated $[R_i(G/\Delta'_i)]$-modules. Then ϕ_i is a morphism $\mathrm{Inf}_{G/\Delta'_i}^G(k\widehat{\otimes}_{R_i}M_i^\bullet) \to \mathrm{Inf}_{G/\Delta}^G V^\bullet$ in $D_{\mathrm{fin}}^-([[kG]])$ and is represented by a pair of morphisms in $C_{\mathrm{fin}}^-([[kG]])$ of the form

(3.7)
$$
\begin{array}{ccc}
 & Z_i^\bullet & \\
 & {}^{\phi_{i1}}\swarrow \quad \searrow {}^{\phi_{i2}} & \\
\mathrm{Inf}_{G/\Delta'_i}^G(k\widehat{\otimes}_{R_i}M_i^\bullet) & & \mathrm{Inf}_{G/\Delta}^G V^\bullet
\end{array}
$$

where ϕ_{i1} is a quasi-isomorphism. There exists an integer n such that for all $j < n$, $\mathrm{H}^j(V^\bullet) = 0$, $\mathrm{H}^j(Z_i^\bullet) = 0$ and $\mathrm{H}^j(M_i^\bullet) = 0$. It follows from Lemma 3.9 that we can divide $Z_i^\bullet$ (resp. $M_i^\bullet$) by a suitable acyclic subcomplex to be able to assume that $Z_i^\bullet$ (resp. $M_i^\bullet$) in (3.7) is bounded and its terms have finite k-length. Since Δ'_i acts trivially on all the terms of $V^\bullet$ and $M_i^\bullet$, there exists an open (and closed) normal subgroup Δ_i of finite index in G with $\Delta_i \subseteq \Delta'_i$ such that Δ_i acts trivially on all the terms of $V^\bullet$, $Z_i^\bullet$ and $M_i^\bullet$. Thus (3.7) defines a morphism in $D_{\mathrm{fin}}^-([k(G/\Delta_i)])$. By Remark 3.5, we can now replace $M_i^\bullet$ by a bounded above complex $N_i^\bullet$ of abstractly free finitely generated $[R_i(G/\Delta_i)]$-modules. By Lemma 14.1, the resulting morphism $\psi_i : k\widehat{\otimes}_{R_i}N_i^\bullet \to \mathrm{Inf}_{G/\Delta}^{G/\Delta_i} V^\bullet$ in $D_{\mathrm{fin}}^-([k(G/\Delta_i)])$ can be taken to be a morphism in $K_{\mathrm{fin}}^-([k(G/\Delta_i)])$. This proves part (i).

In part (ii), the morphism ψ_2 is represented by a morphism $\psi_2 : k\widehat{\otimes}_{R_2}N_2^\bullet \to \mathrm{Inf}_{G/\Delta}^{G/\Delta_2} V^\bullet$ in $C_{\mathrm{fin}}^-([k(G/\Delta_2)])$. The morphism ρ is represented by a pair of morphisms in $C_{\mathrm{fin}}^-([[R_1G]])$

(3.8)
$$
\begin{array}{ccc}
 & T^\bullet & \\
 & {}^{\rho_1}\swarrow \quad \searrow {}^{\rho_2} & \\
\mathrm{Inf}_{G/\Delta_2}^G(R_1\widehat{\otimes}_{R_2}N_2^\bullet) & & \mathrm{Inf}_{G/\Delta_1}^G N_1^\bullet
\end{array}
$$

where ρ_1 is a quasi-isomorphism. There exists an integer n' such that for all $j < n'$, $\mathrm{H}^j(V^\bullet) = 0$, $\mathrm{H}^j(T^\bullet) = 0$, and $\mathrm{H}^j(N_i^\bullet) = 0$ ($i = 1, 2$). It follows from Lemma 3.9 that we can divide $T^\bullet$ by a suitable acyclic subcomplex to be able to assume that $T^\bullet$ in (3.7) is bounded and its terms have finite k-length. Since $\Delta_1 \cap \Delta_2$ acts trivially on all the terms of $V^\bullet$ (resp. $N_i^\bullet$ ($i = 1, 2$)), there exists an open (and closed) normal subgroup $\widetilde{\Delta}_2$ of finite index in G with $\widetilde{\Delta}_2 \subseteq \Delta_1 \cap \Delta_2$ such that $\widetilde{\Delta}_2$ acts trivially on all the terms of $V^\bullet$, $T^\bullet$, and $N_i^\bullet$ ($i = 1, 2$). Thus ψ_2 (resp. (3.7)) defines a morphism in $D_{\mathrm{fin}}^-([k(G/\widetilde{\Delta}_2)])$ (resp. $D_{\mathrm{fin}}^-([R_1(G/\widetilde{\Delta}_2)])$). By Remark 3.5, we can now replace $N_2^\bullet$ by a bounded above complex $\widetilde{N}_2^\bullet$ of abstractly free finitely generated $[R_2(G/\widetilde{\Delta}_2)]$-modules. By Lemma 14.1,

the resulting morphisms $\widetilde{\psi}_2 : k\widehat{\otimes}_{R_2}\widetilde{N}_2^\bullet \to \mathrm{Inf}_{G/\Delta}^{G/\widetilde{\Delta}_2} V^\bullet$ in $D_{\mathrm{fin}}^-([k(G/\widetilde{\Delta}_2)])$ (resp. $\tilde{\rho} : R_1\widehat{\otimes}_{R_2}\widetilde{N}_2^\bullet \to \mathrm{Inf}_{G/\Delta_1}^{G/\widetilde{\Delta}_2} N_1^\bullet$ in $D_{\mathrm{fin}}^-([R_1(G/\widetilde{\Delta}_2)])$) can be taken to be morphisms in $K_{\mathrm{fin}}^-([k(G/\widetilde{\Delta}_2)])$ (resp. $K_{\mathrm{fin}}^-([R_1(G/\widetilde{\Delta}_2)])$). It is obvious that the equality $\mathrm{Inf}_{G/\Delta_1}^{G/\widetilde{\Delta}_2}(\psi_1)\,(k\widehat{\otimes}_{R_1}\tilde{\rho}) = \widetilde{\psi}_2$ is preserved. This proves part (ii) and completes the proof of Corollary 3.11. $\qquad\square$

4. Endomorphisms.

DEFINITION 4.1. — Define $\widehat{F}_1$ (resp. F_1) to be the functor from $\widehat{\mathcal{C}}$ (resp. $\mathcal{C}$) to the category Sets which sends R to the set $\widehat{F}_1(R)$ (resp. $F_1(R)$) of isomorphism classes in $D^-([[RG]])$ of quasi-lifts of $V^\bullet$ which are bounded above complexes of topologically free pseudocompact $[[RG]]$-modules.

LEMMA 4.2. — *The natural transformation $\widehat{F}_1 \to \widehat{F}$ (resp. $F_1 \to F$) is an isomorphism of functors.*

Proof. — Suppose $(M^\bullet, \phi)$ is a quasi-lift of $V^\bullet$ over some ring $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$ (resp. $R \in \mathrm{Ob}(\mathcal{C})$). By Lemma 2.9, we can assume that all the terms of $M^\bullet$ are topologically free pseudocompact R-modules. Since the category of pseudocompact $[[RG]]$-modules has enough projectives, which are direct summands of topologically free pseudocompact $[[RG]]$-modules, there is a complex $M_1^\bullet$ of topologically free pseudocompact $[[RG]]$-modules which is bounded above and isomorphic to $M^\bullet$ in $D^-([[RG]])$. Therefore the original lift $(M^\bullet, \phi)$ is isomorphic to $(M_1^\bullet, \phi_1)$ for a suitable choice of ϕ_1. This proves that the natural transformation $\widehat{F}_1 \to \widehat{F}$ (resp. $F_1 \to F$) is an isomorphism of functors. $\qquad\square$

PROPOSITION 4.3. — *Suppose $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet) = k$. Then $\mathrm{Hom}_{D^-([[RG]])}(M^\bullet, M^\bullet) = R$ for every quasi-lift $(M^\bullet, \phi)$ of $V^\bullet$ over an Artinian object $R \in \mathrm{Ob}(\mathcal{C})$.*

Proof. — By Lemma 4.2, we can reduce to the case in which the terms of $M^\bullet$ are topologically free pseudocompact $[[RG]]$-modules. By the dual of [17, Prop. I.4.7], it follows then that

$$\mathrm{Hom}_{D^-([[RG]])}(M^\bullet, M^\bullet) = \mathrm{Hom}_{K^-([[RG]])}(M^\bullet, M^\bullet).$$

Let now R' be in $\mathcal{C}$ so that $\pi : R \to R'$ is a small extension (i.e. $\mathrm{Ker}(\pi)$ is a principal ideal annihilated by m_R) with $\mathrm{Ker}(\pi) = tR \cong k$. Suppose that $\mathrm{Hom}_{K^-([[R'G]])}(M'^\bullet, M'^\bullet) = R'$, whenever $M'^\bullet$ is a quasi-lift of $V^\bullet$ over R' which is a bounded above complex of topologically free pseudocompact

$[[R'G]]$-modules. Let $\alpha \in \mathrm{Hom}_{C^-([[RG]])}(M^\bullet, M^\bullet)$. Then α induces a map $\alpha' \in \mathrm{Hom}_{C^-([[R'G]])}(M'^\bullet, M'^\bullet)$ when $M'^\bullet = R' \widehat{\otimes}_{R,\pi} M^\bullet$. By induction, there is a scalar $\lambda' \in R'$ such that α' is homotopic to the map $\mu_{\lambda'}$ which is multiplication by λ'. Choose $\lambda \in R$ so that $\pi(\lambda) = \lambda'$. By Lemma 14.3, since $M^\bullet$ is a bounded above complex of topologically free pseudocompact $[[RG]]$-modules, it follows that there is a map $\alpha_1 \in \mathrm{Hom}_{C^-([[RG]])}(M^\bullet, tM^\bullet)$ such that α is homotopic to

$$(4.1) \qquad\qquad \mu_\lambda + \alpha_1.$$

Since $tR \cong k$, and the terms of $M^\bullet$ are topologically free pseudocompact $[[RG]]$-modules, there exists an isomorphism $\tau : tM^\bullet \to k \widehat{\otimes}_R M^\bullet$ in $C^-([[RG]])$. Let $\phi_1 : M^\bullet \to k \widehat{\otimes}_R M^\bullet$ be the natural morphism in $C^-([[RG]])$, which is surjective on terms. We obtain an isomorphism

$$\mathrm{Hom}_{K^-([[RG]])}(M^\bullet, tM^\bullet) \longrightarrow \mathrm{Hom}_{K^-([[kG]])}(k\widehat{\otimes}_R M^\bullet, k\widehat{\otimes}_R M^\bullet)$$

which sends the homotopy class of f to the homotopy class of g when $g \phi_1 = \tau f$. Suppose α_1 is sent to β_1 by this isomorphism. Then, by assumption, β_1 is homotopic to multiplication by a scalar in k, say $\bar\lambda_1$. If $\bar\lambda_1$ corresponds to $t\lambda_1$ under the isomorphism $k \cong tR$, then it follows that α_1 is homotopic to multiplication by $t\lambda_1$. Hence Proposition 4.3 follows from (4.1). □

5. Schlessinger's criteria.

In this section we will prove:

PROPOSITION 5.1. — *Schlessinger's criteria* (H1) *and* (H2) (*see* [25, Thm. 2.11]) *are always satisfied for* $F_{\mathcal{D}}$. *In case* $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet) = k$, (H4) *is also satisfied.*

The following remark will be useful in various proofs.

Remark 5.2. — Suppose $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$. Let $M^\bullet$ and $N^\bullet$ be two bounded above complexes of pseudocompact $[[RG]]$-modules, and let $f : M^\bullet \to N^\bullet$ be a morphism in $C^-([[RG]])$. Let $P^\bullet$ be a bounded above complex of topologically free pseudocompact $[[RG]]$-modules so that $f : P^\bullet \to N^\bullet$ is a quasi-isomorphism in $C^-([[RG]])$ which is surjective on terms. Then the mapping cone $C^\bullet$ of $T^{-1}(P^\bullet) \xrightarrow{\mathrm{id}} T^{-1}(P^\bullet)$ is an acyclic complex, and there is a morphism $\pi : C^\bullet \to N^\bullet$ in $C^-([[RG]])$ which is surjective on terms.

Define $g : M^\bullet \oplus C^\bullet \to N^\bullet$ by $g = (f, \pi)$, and define $s : M^\bullet \to M^\bullet \oplus C^\bullet$ by $s = \binom{\mathrm{id}_{M^\bullet}}{0}$. Then g is surjective on terms, s is a quasi-isomorphism and $g\,s = f$.

Suppose there is a surjective morphism $R_1 \to R$ in $\widehat{\mathcal{C}}$, and there is a bounded above complex $X^\bullet$ of topologically free pseudocompact $[[R_1G]]$-modules such that $M^\bullet = R \widehat{\otimes}_{R_1} X^\bullet$. Since $[[RG]] = R \widehat{\otimes}_{R_1} [[R_1G]]$, there exists a bounded above complex $Q^\bullet$ of topologically free pseudocompact $[[R_1G]]$-modules with $P^\bullet = R \widehat{\otimes}_{R_1} Q^\bullet$. Hence $C^\bullet = R \widehat{\otimes}_{R_1} D^\bullet$, where $D^\bullet$ is the mapping cone of $T^{-1}(Q^\bullet) \xrightarrow{\mathrm{id}} T^{-1}(Q^\bullet)$, and $M^\bullet \oplus C^\bullet = R \widehat{\otimes}_{R_1} (X^\bullet \oplus Q^\bullet)$.

Suppose A, B, C are Artinian objects in $\mathrm{Ob}(\mathcal{C})$ and that we have a diagram in $\mathcal{C}$

$$
\begin{array}{ccc}
A & & B \\
& \searrow^{\alpha} \quad \swarrow_{\beta} & \\
& C &
\end{array}
$$
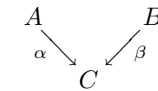
Let D be the pullback $D = A \times_C B = \{(a,b) \in A \times B \mid \alpha(a) = \beta(b)\}$. Consider the natural map

$$\chi_{\mathcal{D}} : F_{\mathcal{D}}(D) \longrightarrow F_{\mathcal{D}}(A) \times_{F_{\mathcal{D}}(C)} F_{\mathcal{D}}(B).$$

LEMMA 5.3. — *If β is surjective, then $\chi_{\mathcal{D}}$ is surjective.*

Proof. — Suppose $(X_A^\bullet, \xi_A) \in F_{\mathcal{D}}(A)$ and $(X_B^\bullet, \xi_B) \in F_{\mathcal{D}}(B)$ such that there exists an isomorphism

$$\tau : C \widehat{\otimes}_B^L X_B^\bullet \to C \widehat{\otimes}_A^L X_A^\bullet$$

in $D^-([[CG]])$ with $\xi_A (k\widehat{\otimes}_C^L \tau) = \xi_B$. By Corollary 3.10, there exists a closed normal subgroup Δ of finite index in G so that we can assume the following. The complex $X_A^\bullet$ (resp. $X_B^\bullet$) is a bounded above complex of abstractly free finitely generated $[A(G/\Delta)]$-modules (resp. $[B(G/\Delta)]$-modules), τ is given by a quasi-isomorphism in $C^-([C(G/\Delta)])$, and ξ_A (resp. ξ_B) is given by a quasi-isomorphism in $C^-([k(G/\Delta)])$. By Remark 5.2, we can add to $X_B^\bullet$ an acyclic complex of abstractly free finitely generated $[B(G/\Delta)]$-modules to be able to assume that τ is surjective on terms. By Lemma 14.4, there exists a bounded above complex $L_B^\bullet$ of abstractly free finitely generated $[B(G/\Delta)]$-modules and a quasi-isomorphism $\psi : X_B^\bullet \to L_B^\bullet$ in $C^-([B(G/\Delta)])$ so that there is an isomorphism $\pi : C \otimes_B L_B^\bullet \to C \otimes_A X_A^\bullet$ in $C^-([C(G/\Delta)])$ with

$\pi(C \otimes_B \psi) = \tau$. We replace $X_B^\bullet$ by $L_B^\bullet$, τ by π and ξ_B by $\xi_A (k \otimes_C \pi)$ to have

$$(5.1) \qquad C \otimes_A X_A^\bullet = \pi(C \otimes_B X_B^\bullet) = X_C^\bullet$$

in $C^-([C(G/\Delta)])$. Using (5.1), we define $X_D^\bullet = X_A^\bullet \times_{X_C^\bullet} X_B^\bullet$ to be the complex whose i-th term is $X_D^i = X_A^i \times_{X_C^i} X_B^i$ and whose boundary maps are $\delta_{X_D}^i(x^i, y^i) = (\delta_{X_A}^i(x^i), \delta_{X_B}^i(y^i))$. Then $X_D^\bullet$ is a bounded above complex of abstractly free finitely generated $[D(G/\Delta)]$-modules which has finite pseudocompact D-tor dimension. Moreover, $C \otimes_D X_D^\bullet = C \otimes_A X_A^\bullet$, and we can define $\xi_D : k \otimes_D X_D^\bullet \to V^\bullet$ by $\xi_D = \xi_A$. It follows that $(X_D^\bullet, \xi_D)$ is an element in $F(D)$. Hence $\chi_\mathcal{D}$ is surjective in case $F_\mathcal{D} = F$.

In case $F_\mathcal{D} = F^{\mathrm{fl}}$, we additionally have to show that $X_D^\bullet$ defines an element in $F^{\mathrm{fl}}(D)$, i.e. its cohomology groups are topologically free pseudocompact D-modules. By (5.1), it follows that, for all integers n, X_A^n and X_B^n are abstractly free of the same finite rank over A, resp. B. Additionally, by Lemma 2.11, $\mathrm{H}^n(X_A^\bullet)$ and $\mathrm{H}^n(X_B^\bullet)$ are abstractly free of the same finite rank over A, resp. B. Since we have arranged that all terms of $X_A^\bullet$ (resp. $X_B^\bullet$) are abstractly free finitely generated A-modules (resp. B-modules), we obtain that the terms of $X_A^\bullet$ (resp. $X_B^\bullet$) split completely as in (2.1). Hence, since $D = A \times_C B$, it follows that $\mathrm{H}^n(X_D^\bullet) = \mathrm{H}^n(X_A^\bullet) \times_{\mathrm{H}^n(X_C^\bullet)} \mathrm{H}^n(X_B^\bullet)$. This completes the proof of Lemma 5.3. $\qquad\square$

In view of [25, Thm. 2.11], Proposition 5.1 now follows from the next result.

LEMMA 5.4. — *If β is surjective, and either $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet) = k$ or $C = k$, then $\chi_\mathcal{D}$ is injective.*

Proof. — Since F^{fl} is a subfunctor of F by Proposition 2.12, it is enough to show the statement of Lemma 5.4 in case $F_\mathcal{D} = F$. Suppose $(X_D^\bullet, \xi)$ and $(Z_D^\bullet, \zeta)$ are two elements in $F(D)$ such that there is an isomorphism $\tau_A : A \widehat{\otimes}_D^L Z_D^\bullet \to A \widehat{\otimes}_D^L X_D^\bullet$ (resp. $\tau_B : B \widehat{\otimes}_D^L Z_D^\bullet \to B \widehat{\otimes}_D^L X_D^\bullet$) in $D^-([[AG]])$ (resp. $D^-([[BG]])$) with $\xi(k \widehat{\otimes}_A^L \tau_A) = \zeta$ (resp. $\xi(k \widehat{\otimes}_B^L \tau_B) = \zeta$) in $D^-([[kG]])$. In other words $A \widehat{\otimes}_D^L Z_D^\bullet$ and $A \widehat{\otimes}_D^L X_D^\bullet$ (resp. $B \widehat{\otimes}_D^L Z_D^\bullet$ and $B \widehat{\otimes}_D^L X_D^\bullet$) are isomorphic as quasi-lifts of $V^\bullet$ over A (resp. B). Consider $\varphi_C : C \widehat{\otimes}_D^L Z_D^\bullet \to C \widehat{\otimes}_D^L Z_D^\bullet$ in $D^-([[CG]])$, defined by $\varphi_C = (C \widehat{\otimes}_A^L \tau_A)^{-1}(C \widehat{\otimes}_B^L \tau_B)$. If $C = k$, then

$$\varphi_k = (k \widehat{\otimes}_A^L \tau_A)^{-1}(k \widehat{\otimes}_B^L \tau_B) = (\zeta^{-1}\xi)(\xi^{-1}\zeta) = \mathrm{id}_{k \widehat{\otimes}_D^L Z_D^\bullet}$$

in $D^-([[kG]])$. If $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet) = k$, then, by Proposition 4.3,

$$\mathrm{Hom}_{D^-([[CG]])}(C \widehat{\otimes}_D^L Z_D^\bullet, C \widehat{\otimes}_D^L Z_D^\bullet) = C.$$

Hence, in either case there exists a unit $\alpha_C \in C$, with image 1 in k, so that φ_C is multiplication by α_C in $D^-([[CG]])$. By Corollary 3.10, we can find a closed normal subgroup Δ of finite index in G so that we can assume the following.

(i) The complex $V^\bullet$ is a bounded above complex of abstractly free finitely generated $[k(G/\Delta)]$-modules, and $X_D^\bullet$ and $Z_D^\bullet$ are bounded above complexes of abstractly free finitely generated $[D(G/\Delta)]$-modules.

(ii) The morphisms $\xi : k \otimes_D X_D^\bullet \to V^\bullet$ and $\zeta : k \otimes_D Z_D^\bullet \to V^\bullet$ are given by quasi-isomorphisms in $C^-([k(G/\Delta)])$.

(iii) The morphism $\tau_A : A \otimes_D Z_D^\bullet \to A \otimes_D X_D^\bullet$ (resp. $\tau_B : B \otimes_D Z_D^\bullet \to B \otimes_D X_D^\bullet$) is given by a quasi-isomorphism in $C^-([A(G/\Delta)])$ (resp. in $C^-([B(G/\Delta)])$) such that $\xi(k \otimes_A \tau_A) = \zeta$ (resp. $\xi(k \otimes_B \tau_B) = \zeta$) in $K^-([k(G/\Delta)])$.

Since β is surjective, $D \to A$ is also surjective. By Remark 5.2, we can add to $Z_D^\bullet$ an acyclic complex of abstractly free finitely generated $[D(G/\Delta)]$-modules to be able to assume that τ_A is surjective on terms. By Lemma 14.4, we can adjust $Z_D^\bullet$ so that τ_A is an isomorphism in $C^-([A(G/\Delta)])$. These adjustments preserve (i) through (iii). Since multiplication by α_C is a morphism in $C_{\mathrm{fin}}^-([C(G/\Delta)])$ which commutes with all morphisms with suitable domains and codomains, it follows that $\varphi_C = (C \otimes_A \tau_A)^{-1}(C \otimes_B \tau_B)$ is given by a quasi-isomorphism in $C^-([C(G/\Delta)])$ such that φ_C is homotopic to multiplication by α_C. By Lemma 14.3, we can lift this homotopy to one from $B \otimes_D Z_D^\bullet$ to itself. Hence we can adjust τ_B so as to be able to assume that φ_C is exactly equal to multiplication by α_C in $C^-([C(G/\Delta)])$. Since β is surjective, we can lift α_C to a unit scalar $\alpha_B \in B$. We replace τ_B by $\alpha_B^{-1}\tau_B$. Thus we obtain diagram (5.2) (see next page) in $C^-([D(G/\Delta)])$, where $\tau_{A,C} = C \otimes \tau_A$, $\tau_{B,C} = C \otimes \tau_B$, and $\varphi_C = \tau_{A,C}^{-1}\tau_{B,C}$ is the identity on $C \otimes_D Z_D^\bullet$ in $C^-([C(G/\Delta)])$.

Claim. — We have $Z_D^\bullet = (A \otimes_D Z_D^\bullet) \times_{C \otimes_D Z_D^\bullet} (B \otimes_D Z_D^\bullet)$ and $X_D^\bullet = (A \otimes_D X_D^\bullet) \times_{C \otimes_D X_D^\bullet} (B \otimes_D X_D^\bullet)$ in $C^-([D(G/\Delta)])$. The morphism $\tau : Z_D^\bullet \to X_D^\bullet$ defined by $\tau(z_A, z_B) = (\tau_A(z_A), \tau_B(z_B))$ is a quasi-isomorphism in $C^-([D(G/\Delta)])$ such that $\xi(k \otimes_D \tau_Z) = \zeta$ in $K^-([k(G/\Delta)])$.

Proof of claim. — We first show that

$$Z_D^\bullet = (A \otimes_D Z_D^\bullet) \times_{C \otimes_D Z_D^\bullet} (B \otimes_D Z_D^\bullet)$$

in $C^-(D(G/\Delta))$. Define the map $\gamma^i : Z_D^i \to (A \otimes_D Z_D^i) \times_{C \otimes_D Z_D^i} (B \otimes_D Z_D^i)$

$$(5.2)$$

by $\gamma^i(z^i) = (1_A \otimes z^i, 1_B \otimes z^i)$. Since Z_D^i is an abstractly free finitely generated $[D(G/\Delta)]$-module, γ^i is an isomorphism of $[D(G/\Delta)]$-modules. Furthermore, $\gamma = (\gamma^i)$ respects the boundary maps of $Z_D^\bullet$, resp. of $(A \otimes_D Z_D^\bullet) \times_{C \otimes_D Z_D^\bullet} (B \otimes_D Z_D^\bullet)$. Similarly, one shows that $X_D^\bullet$ is the pullback $(A \otimes_D X_D^\bullet) \times_{C \otimes_D X_D^\bullet} (B \otimes_D X_D^\bullet)$. Since $\xi(k \otimes_A \tau_A) = \zeta = \xi(k \otimes_B \tau_B)$ in $K^-([k(G/\Delta)])$, it also follows that $\xi(k \otimes_D \tau_D) = \zeta$ in $K^-([k(G/\Delta)])$.

We now show that τ is a quasi-isomorphism in $C^-([D(G/\Delta)])$. By Remark 5.2, we can add a suitable acyclic complex of abstractly free finitely generated $[D(G/\Delta)]$-modules to $Z_D^\bullet$ to be able to assume that $\tau = (\tau_A, \tau_B) : Z_D^\bullet \to X_D^\bullet$ is surjective on terms. Note that this adjustment preserves the isomorphism class of the quasi-lift $(Z_D^\bullet, \zeta)$. The complex $K_D^\bullet$ formed by the kernels of the terms of τ is then a complex of abstractly free finitely generated $[D(G/\Delta)]$-modules, since the terms of $Z_D^\bullet$ and of $X_D^\bullet$ have these properties. Furthermore, the complexes $K_A^\bullet = A \otimes_D K_D^\bullet$ and $K_B^\bullet = B \otimes_D K_D^\bullet$ are acyclic, because τ_A and τ_B are quasi-isomorphisms which are surjective on terms. We see that $K_D^\bullet = K_A^\bullet \times_{K_C^\bullet} K_B^\bullet$. We can form compatible splittings of $K_A^\bullet$ and $K_B^\bullet$ in the following way. Choose a splitting of $K_A^\bullet$. This induces a splitting of $K_C^\bullet$, which can be lifted to one of $K_B^\bullet$ because $\beta : B \to C$ is surjective. The resulting splittings of $K_A^\bullet$ and $K_B^\bullet$ now give a splitting of $K_D^\bullet$, proving that $K_D^\bullet$ is acyclic. Hence τ is a quasi-isomorphism in $C^-([D(G/\Delta)])$, which completes the proof of Lemma 5.4. $\qquad\square$

6. The tangent space.

In this section we will prove (H3) of Schlessinger's criteria in Proposition 6.4. This implies that the functor $F_\mathcal{D}$ is pro-representable by [25, Thm. 2.11].

Let $k[\varepsilon]$, where $\varepsilon^2 = 0$, denote the ring of dual numbers over k. Recall that $t_{F_\mathcal{D}} = F_\mathcal{D}(k[\varepsilon])$ is called the tangent space to $F_\mathcal{D}$. By [25, Lemma 2.10], $t_{F_\mathcal{D}}$ is a vector space over k. We first determine the k-vector space structure in case $F_\mathcal{D} = F$.

LEMMA 6.1. — *There is a k-vector space isomorphism*

$$h : t_F = F(k[\varepsilon]) \longrightarrow \mathrm{Hom}_{D^-([[kG]])}\big(V^\bullet, T(V^\bullet)\big) = \mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet, V^\bullet)$$

where, as before, T is the translation functor.

Proof. — By Proposition 2.12, we may assume that $V^\bullet$ is a bounded above complex of topologically free pseudocompact $[[kG]]$-modules. Suppose $(M^\bullet, \phi)$ is a quasi-lift of $V^\bullet$ over $k[\varepsilon]$. By Lemma 4.2, we can assume that $M^\bullet$ is a bounded above complex of topologically free pseudocompact $[[k[\varepsilon]G]]$-modules. We have a short exact sequence

$$0 \to \varepsilon M^\bullet \xrightarrow{\iota} M^\bullet \xrightarrow{\pi} M^\bullet / \varepsilon M^\bullet \to 0$$

in $C^-([[k[\varepsilon]G]])$. The mapping cone of ι is $C(\iota)^\bullet = T(\varepsilon M^\bullet) \oplus M^\bullet$ with i-th differential

$$\delta_{C(\iota)}^i = \begin{pmatrix} -\delta_M^{i+1} & 0 \\ \iota^{i+1} & \delta_M^i \end{pmatrix}.$$

We obtain a triangle in $K^-([[k[\varepsilon]G]])$

$$(6.1) \qquad \varepsilon M^\bullet \xrightarrow{\iota} M^\bullet \xrightarrow{g} C(\iota)^\bullet \xrightarrow{f} T(\varepsilon M^\bullet)$$

where $g^i(b) = (0, b)$ and $f^i(a, b) = -a$. We define two morphisms in $C^-([[k[\varepsilon]G]])$

$$(0, \pi) : C(\iota)^\bullet = T(\varepsilon M^\bullet) \oplus M^\bullet \longrightarrow M^\bullet / \varepsilon M^\bullet,$$

$$\psi : \varepsilon M^\bullet \longrightarrow M^\bullet / \varepsilon M^\bullet,$$

by $(0, \pi)^i(a, b) = \pi^i(b)$ and $\psi^i(\varepsilon x) = \pi^i(x)$. The kernel of $(0, \pi)$ is the mapping cone of $\varepsilon M^\bullet \xrightarrow{\mathrm{id}} \varepsilon M^\bullet$ which is acyclic; hence $(0, \pi)$ is a quasi-isomorphism. The morphism ψ is an isomorphism of complexes with inverse ψ^{-1} given by $(\psi^{-1})^i(\pi^i(x)) = \varepsilon x$. The mapping cone of f from (6.1) is

$$C(f)^\bullet = T^2(\varepsilon M^\bullet) \oplus T(M^\bullet) \oplus T(\varepsilon M^\bullet)$$

with i-th differential

$$\delta^i_{C(f)} = \begin{pmatrix} \delta^{i+2}_M & 0 & 0 \\ -\iota^{i+2} & -\delta^{i+1}_M & 0 \\ -1 & 0 & -\delta^{i+1}_M \end{pmatrix}.$$

Then we have a quasi-isomorphism in $C^-([[k[\varepsilon]G]])$

$$(6.3) \qquad\qquad \rho : C(f)^\bullet \longrightarrow T(M^\bullet)$$

given by $\rho^i(a,b,c) = b - \iota^{i+1}(c)$. We get a triangle in $K^-([[k[\varepsilon]G]])$

$$(6.4) \qquad \begin{array}{ccccccc} C(\iota)^\bullet & \xrightarrow{f} & T(\varepsilon M^\bullet) & \longrightarrow & C(f)^\bullet & \longrightarrow & T(C(\iota)^\bullet) \\ {\scriptstyle (0,\pi)}\downarrow & & \| & & \rho\downarrow & & \downarrow{\scriptstyle T(0,\pi)} \\ M^\bullet/\varepsilon M^\bullet & & T(\varepsilon M^\bullet) & & T(M^\bullet) & & T(M^\bullet/\varepsilon M^\bullet) \end{array}$$

where the downward arrows are quasi-isomorphisms in $C^-([[k[\varepsilon]G]])$. Hence the diagram

$$(6.5) \qquad \begin{array}{ccc} & C(\iota)^\bullet & \\ {\scriptstyle (0,\pi)}\swarrow & & \searrow{\scriptstyle f} \\ M^\bullet/\varepsilon M^\bullet & & T(\varepsilon M^\bullet) \end{array}$$

defines a morphism $\widehat{f} : M^\bullet/\varepsilon M^\bullet \to T(\varepsilon M^\bullet)$ in $D^-([[k[\varepsilon]G]])$. Because of (6.4), we obtain a triangle in $D^-([[k[\varepsilon]G]])$:

$$M^\bullet/\varepsilon M^\bullet \xrightarrow{\widehat{f}} T(\varepsilon M^\bullet) \longrightarrow T(M^\bullet) \longrightarrow T(M^\bullet/\varepsilon M^\bullet).$$

Using the isomorphism $\phi : M^\bullet/\varepsilon M^\bullet \to V^\bullet$ in $D^-([[kG]])$, we obtain a morphism

$$\widehat{f}_1 \in \operatorname{Hom}_{D^-([[k[\varepsilon]G]])}(V^\bullet, T(V^\bullet))$$

associated to $\widehat{f}$, namely $\widehat{f}_1 = \phi' \widehat{f} \phi^{-1}$, where $\phi' = T(\phi)T(\psi)$ and ψ is as in (6.2). We get an association $\widehat{h}$ defined by

$$(6.6) \quad \widehat{h}: \quad F(k[\varepsilon]) \longrightarrow \operatorname{Hom}_{D^-([[k[\varepsilon]G]])}(V^\bullet, T(V^\bullet)), \quad (M^\bullet, \phi) \longmapsto \widehat{f}_1.$$

    Claim 1. — The association $\widehat{h}$ is a well-defined injective set map.

    Proof of Claim 1. — Let $(N^\bullet, \theta)$ be another quasi-lift of $V^\bullet$ over $k[\varepsilon]$ such that $N^\bullet$ is a bounded above complex of topologically free

pseudocompact $[[k[\varepsilon]G]]$-modules. Let $\widehat{g}$ be the morphism $\widehat{g} : N^\bullet/\varepsilon N^\bullet \to T(\varepsilon N^\bullet)$ in $D^-([[k[\varepsilon]G]])$ which is defined analogously to $\widehat{f}$ in (6.5). If $\tau : M^\bullet \to N^\bullet$ is an isomorphism of quasi-lifts, i.e. τ is an isomorphism in $D^-([[k[\varepsilon]G]])$ with $\theta(k \otimes \tau) = \phi$, then it follows from the definition of $\widehat{g}$ that $T(\tau)\widehat{f} = \widehat{g}(k \otimes \tau)$. A straight forward calculation now shows that $\widehat{h}((M^\bullet, \phi)) = \widehat{h}((N^\bullet, \theta))$, which means that $\widehat{h}$ is well-defined. To prove that $\widehat{h}$ is injective, suppose that $\widehat{h}((M^\bullet, \phi)) = \widehat{f}_1 = \widehat{h}((N^\bullet, \theta))$. By the triangle axiom (TR1) (see e.g. [17, p. 20–23]), there exists a complex $\widehat{M}^\bullet$ in $D^-([[k[\varepsilon]G]])$ so that we have a triangle

$$V^\bullet \xrightarrow{\widehat{f}_1} T(V^\bullet) \longrightarrow T(\widehat{M}^\bullet) \longrightarrow T(V^\bullet)$$

in $D^-([[k[\varepsilon]G]])$. We obtain the following diagram of triangles

$$\begin{array}{ccccccc} M^\bullet/\varepsilon M^\bullet & \xrightarrow{\widehat{f}} & T(\varepsilon M^\bullet) & \longrightarrow & T(M^\bullet) & \longrightarrow & T(M^\bullet/\varepsilon M^\bullet) \\ \phi\downarrow & & \downarrow{\scriptstyle \phi'} & & & & \downarrow{\scriptstyle T(\phi)} \\ V^\bullet & \xrightarrow{\widehat{f}_1} & T(V^\bullet) & \longrightarrow & T(\widehat{M}^\bullet) & \longrightarrow & T(V^\bullet) \\ \theta\uparrow & & \uparrow{\scriptstyle \theta'} & & & & \uparrow{\scriptstyle T(\theta)} \\ N^\bullet/\varepsilon N^\bullet & \xrightarrow{\widehat{g}} & T(\varepsilon M^\bullet) & \longrightarrow & T(N^\bullet) & \longrightarrow & T(N^\bullet/\varepsilon N^\bullet) \end{array}$$

where $\phi' = T(\phi)T(\psi)$, θ' is defined analogously to ϕ', and the downward, resp. upward, arrows are isomorphisms in $D^-([[k[\varepsilon]G]])$. By the triangle axiom (TR3) and by [17, Prop. I.1.1], there exists an isomorphism $\phi'' : T(M^\bullet) \to T(\widehat{M}^\bullet)$ (resp. $\theta'' : T(N^\bullet) \to T(\widehat{M}^\bullet)$) in $D^-([[k[\varepsilon]G]])$, so that (ϕ, ϕ', ϕ'') (resp. $(\theta, \theta', \theta'')$) is an isomorphism of triangles. Thus we obtain a commutative diagram

$$\begin{array}{ccccc} T(M^\bullet) & \xrightarrow{\phi''} & T(\widehat{M}^\bullet) & \xleftarrow{\theta''} & T(N^\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ T(M^\bullet/\varepsilon M^\bullet) & \xrightarrow{T(\phi)} & T(V^\bullet) & \xleftarrow{T(\theta)} & T(N^\bullet/\varepsilon N^\bullet) \end{array}$$

which shows that $(M^\bullet, \phi)$ and $(N^\bullet, \theta)$ are isomorphic quasi-lifts of $V^\bullet$ over $k[\varepsilon]$. This proves Claim 1.

    Since $V^\bullet$ and $M^\bullet$ were chosen to be bounded above complexes of topologically free pseudocompact modules, ϕ can be represented by a quasi-isomorphism in $C^-([[k[\varepsilon]G]])$. Hence it also follows that $\phi' = T(\phi)T(\psi)$ is represented by a quasi-isomorphism in $C^-([[k[\varepsilon]G]])$. Thus we can represent

the morphism $\widehat{f}_1 \in \mathrm{Hom}_{D^-([[k[\varepsilon]G]])}(V^\bullet, T(V^\bullet))$ by the following diagram

$$\widehat{f}_1: \quad \begin{array}{ccc} & C(\iota)^\bullet & \\ {}^{(0,\pi)}\nearrow & & \searrow {}^f \\ M^\bullet/\varepsilon M^\bullet & & T(\varepsilon M^\bullet) \\ {}_{\phi}\searrow & & \swarrow {}_{\phi'} \\ V^\bullet & & T(V^\bullet) \end{array}$$

where the arrows are all given by morphisms in $C^-([[k[\varepsilon]G]])$. The restrictions

$$t_1 = \mathrm{Res}_{[[k[\varepsilon]G]]}^{[[kG]]}(\widehat{f}_1) \quad \text{and} \quad t = \mathrm{Res}_{[[k[\varepsilon]G]]}^{[[kG]]}(\widehat{f})$$

are morphisms in $D^-([[kG]])$. They are represented by the following diagram

$$\begin{array}{ccc} & C(\iota)^\bullet & \\ {}^{(0,\pi)}\swarrow & & \searrow {}^f \\ M^\bullet/\varepsilon M^\bullet & \xrightarrow{\ t\ } & T(\varepsilon M^\bullet) \\ {}_{\phi}\searrow & & \searrow {}_{\phi'} \\ V^\bullet & \xrightarrow{\ t_1\ } & T(V^\bullet) \end{array}$$

where the morphisms $(0,\pi)$, f, ϕ and ϕ' are all viewed as morphisms in $C^-([[kG]])$. Since $t_1\phi = \phi't$ in $K^-([[kG]])$, it follows that

$$\widehat{f}_1 - \mathrm{Inf}_{[[kG]]}^{[[k[\varepsilon]G]]} \mathrm{Res}_{[[k[\varepsilon]G]]}^{[[kG]]}(\widehat{f}_1) = \widehat{f}_1 - \mathrm{Inf}_{[[kG]]}^{[[k[\varepsilon]G]]}(t_1)$$

is represented by the diagram

$$(6.7) \qquad \begin{array}{ccc} & C(\iota)^\bullet & \\ {}^{\phi(0,\pi)}\swarrow & & \searrow {}^{\phi'(f-t(0,\pi))} \\ V^\bullet & & T(V^\bullet) \end{array}$$

Claim 2. — Suppose $X^\bullet = k[\varepsilon] \otimes_k V^\bullet$ is the trivial lift over $k[\varepsilon]$. Then

$$\widehat{f}_1 - \mathrm{Inf}_{[[kG]]}^{[[k[\varepsilon]G]]}(t_1) = \widehat{h}(X^\bullet)$$

in $\mathrm{Hom}_{D^-([[k[\varepsilon]G]])}(V^\bullet, T(V^\bullet))$.

Proof of Claim 2. — Since $M^\bullet/\varepsilon M^\bullet$ is a bounded above complex of topologically free pseudocompact $[[kG]]$-modules, and since $(0,\pi)$ is surjective on terms, there exists by Lemma 14.1 a quasi-isomorphism

$$s: M^\bullet/\varepsilon M^\bullet \longrightarrow C(\iota)^\bullet$$

in $C^-([[kG]])$ with $(0,\pi)s = \mathrm{id}_{M^\bullet/\varepsilon M^\bullet}$ in $C^-([[kG]])$. Because ε acts as zero on $M^\bullet/\varepsilon M^\bullet$, $(0,\pi)s = \mathrm{id}_{M^\bullet/\varepsilon M^\bullet}$ as a morphism $C^-([[k[\varepsilon]G]])$. Then $t = fs$ in $C^-([[kG]])$, and, since ε acts as zero on domain and range, $t \in \mathrm{Hom}_{C^-([[k[\varepsilon]G]])}(M^\bullet/\varepsilon M^\bullet, T(\varepsilon M^\bullet))$.

The trivial lift $X^\bullet$ is isomorphic in $D^-([[k[\varepsilon]G]])$ to the complex $L^\bullet = M^\bullet/\varepsilon M^\bullet \oplus \varepsilon M^\bullet$ with ε-action given by $\varepsilon(x,y) = (0, (\psi^{-1})^i(x))$ for all $(x,y) \in L^i$, where ψ is the isomorphism of complexes from (6.2). The i-th differential is $\delta_L^i(x,y) = (\bar{\delta}_M^i(x), \delta_M^i(y))$ where $\bar{\delta}_M^i(\pi^i(\widehat{x})) = \pi^i(\delta_M^i(\widehat{x}))$ for all $\widehat{x} \in M^i$. From the ε-action on $L^\bullet$ it follows that $\varepsilon L^\bullet = \varepsilon M^\bullet$ and $L^\bullet/\varepsilon L^\bullet = M^\bullet/\varepsilon M^\bullet$. Hence the isomorphism $L^\bullet/\varepsilon L^\bullet \to V^\bullet$ in $D^-([[kG]])$ is given by ϕ.

We now show that $\widehat{h}((L^\bullet, \phi)) = \widehat{f}_1 - \mathrm{Inf}_{[[kG]]}^{[[k[\varepsilon]G]]}(t_1)$, by following the definition of $\widehat{h}$ given prior to Claim 1. We use the subscript 0 to describe the respective morphisms for $(L^\bullet, \phi)$. Then

$$\iota_0: \varepsilon L^\bullet = \varepsilon M^\bullet \longrightarrow M^\bullet/\varepsilon M^\bullet \oplus \varepsilon M^\bullet = L^\bullet$$

is given as $\iota_0^i(x) = (0, x)$. The mapping cone $C(\iota_0)^\bullet$ is

$$C(\iota_0)^\bullet = T(\varepsilon L^\bullet) \oplus L^\bullet = T(\varepsilon M^\bullet) \oplus M^\bullet/\varepsilon M^\bullet \oplus \varepsilon M^\bullet$$

with the usual differential, and

$$f_0: C(\iota_0)^\bullet \longrightarrow T(\varepsilon M^\bullet) = T(\varepsilon L^\bullet)$$

is given as $f_0^i(a,b,c) = -a$. The quasi-isomorphism

$$(0,\pi_0): C(\iota_0)^\bullet \longrightarrow M^\bullet/\varepsilon M^\bullet = L^\bullet/\varepsilon L^\bullet$$

is given as $(0,\pi_0)^i(a,b,c) = b$. We now compare the two morphisms $\widehat{f}_0$ and $(f\widehat{-t(0,\pi)})$ in $D^-([[k[\varepsilon]G]])$. They can be represented by the following two diagrams (compare to (6.5))

$$\widehat{f}_0: \quad \begin{array}{ccc} & C(\iota_0)^\bullet & \\ {}^{(0,\pi_0)}\swarrow & & \searrow {}^{f_0} \\ M^\bullet/\varepsilon M^\bullet & & T(\varepsilon M^\bullet) \end{array} \qquad (f\widehat{-t(0,\pi)}): \quad \begin{array}{ccc} & C(\iota)^\bullet & \\ {}^{(0,\pi)}\swarrow & & \searrow {}^{f-t(0,\pi)} \\ M^\bullet/\varepsilon M^\bullet & & T(\varepsilon M^\bullet) \end{array}$$

where the morphisms representing the arrows all lie in $C^-([[k[\varepsilon]G]])$. Define a morphism

$$u: C(\iota_0)^\bullet = T(\varepsilon M^\bullet) \oplus M^\bullet/\varepsilon M^\bullet \oplus \varepsilon M^\bullet \longrightarrow T(\varepsilon M^\bullet) \oplus M^\bullet = C(\iota)^\bullet$$

in $C^-([[k[\varepsilon]G]])$ by $u^i(a,b,c) = (a,c) + s^i(b)$. Then $(0,\pi)u = (0,\pi_0)$ is a quasi-isomorphism, and $(f - t(0,\pi))u = f_0$. Hence by [17, p. 30], we obtain $\widehat{f_0} = (f - \widehat{t(0,\pi)})$ in $D^-([[k[\varepsilon]G]])$. This implies

$$\widehat{h}\big((L^\bullet,\phi)\big) = \phi'\widehat{f_0}\phi^{-1} = \phi'\big(\widehat{f - t(0,\pi)}\big)\phi^{-1} = \widehat{f_1} - \mathrm{Inf}_{[[kG]]}^{[[k[\varepsilon]G]]}(t_1)$$

in $D^-([[k[\varepsilon]G]])$, where the last equality follows from (6.7). Since $(L^\bullet,\phi)$ is isomorphic to the trivial lift $X^\bullet$ over $k[\varepsilon]$, this proves Claim 2.

By Claim 2,

$$\widehat{f_1} = \widehat{h}(X^\bullet) + \mathrm{Inf}_{[[kG]]}^{[[k[\varepsilon]G]]}\mathrm{Res}_{[[k[\varepsilon]G]]}^{[[kG]]}(\widehat{f_1})$$

in $\mathrm{Hom}_{D^-([[k[\varepsilon]G]])}(V^\bullet, T(V^\bullet))$. Since the map $\widehat{h}$ in (6.6) is injective, we obtain an injective map

$$(6.8) \quad \begin{cases} F\big(k[\varepsilon]\big) \to \mathrm{Inf}_{[[kG]]}^{[[k[\varepsilon]G]]}\mathrm{Hom}_{D^-([[kG]])}\big(V^\bullet, T(V^\bullet)\big), \\ (M^\bullet,\phi) \mapsto \mathrm{Inf}_{[[kG]]}^{[[k[\varepsilon]G]]}\mathrm{Res}_{[[k[\varepsilon]G]]}^{[[kG]]}(\widehat{f_1}). \end{cases}$$

Because the inflation map

$$\mathrm{Inf}_{[[kG]]}^{[[k[\varepsilon]G]]} : \mathrm{Hom}_{D^-([[kG]])}\big(V^\bullet, T(V^\bullet)\big) \longrightarrow \mathrm{Hom}_{D^-([[k[\varepsilon]G]])}\big(V^\bullet, T(V^\bullet)\big)$$

is injective, the map (6.8) turns into an injective map

$$h : F\big(k[\varepsilon]\big) \longrightarrow \mathrm{Hom}_{D^-([[kG]])}(V^\bullet, T(V^\bullet)), \quad (M^\bullet,\phi) \longmapsto \mathrm{Res}_{[[k[\varepsilon]G]]}^{[[kG]]}(\widehat{f_1}).$$

Claim 3. — The map h is surjective.

Proof of Claim 3. — Suppose $\alpha \in \mathrm{Hom}_{D^-([[kG]])}(V^\bullet, T(V^\bullet))$. Since we assumed, without loss of generality, that $V^\bullet$ is a bounded above complex of topologically free pseudocompact $[[kG]]$-modules, we can take $\alpha : V^\bullet \to T(V^\bullet)$ to be represented by a morphism in $C^-([[kG]])$. We define $M_\alpha^\bullet = C(T^{-1}(\alpha))^\bullet$, i.e.

$$(6.9) \qquad M_\alpha^\bullet = V^\bullet \oplus V^\bullet \quad \text{with } i\text{-th differential } \delta_{M_\alpha}^i = \begin{pmatrix} \delta_V^i & 0 \\ \alpha^i & \delta_V^i \end{pmatrix},$$

and give it an ε-action by $\varepsilon(a,b) = (0,a)$ for all $(a,b) \in M_\alpha^i$. It follows that for all i, M_α^i is a topologically free pseudocompact, and thus abstractly free, $k[\varepsilon]$-module. Further, $M_\alpha^\bullet/\varepsilon M_\alpha^\bullet = V^\bullet$. Thus, if $M_\alpha^\bullet$ has

finite pseudocompact $k[\varepsilon]$-tor dimension, it follows that $M_\alpha^\bullet$ defines a quasi-lift $(M_\alpha^\bullet, \phi_\alpha)$ of $V^\bullet$ over $k[\varepsilon]$. Following the definition of $h((M_\alpha^\bullet, \phi_\alpha))$, we then see that $h((M_\alpha^\bullet, \phi_\alpha))$ is the morphism in $D^-([[kG]])$ represented by the following diagram

$$\begin{array}{ccc} & T(V^\bullet) \oplus V^\bullet \oplus V^\bullet & \\ {\scriptstyle (0,\mathrm{id}_{V^\bullet},0)} \swarrow & & \searrow {\scriptstyle (-\mathrm{id}_{T(V^\bullet)},0,0)} \\ V^\bullet & & T(V^\bullet) \end{array}$$

where the arrows are morphisms in $C^-([[kG]])$. We now define a morphism

$$s_\alpha : V^\bullet \longrightarrow T(V^\bullet) \oplus V^\bullet \oplus V^\bullet$$

in $C^-([[kG]])$ by $s_\alpha = \begin{pmatrix} -\alpha \\ \mathrm{id}_{V^\bullet} \\ 0 \end{pmatrix}$. Then $(0, \mathrm{id}_{V^\bullet}, 0)\, s_\alpha = \mathrm{id}_{V^\bullet}$. Hence

$$h((M_\alpha^\bullet, \phi_\alpha)) = h((M_\alpha^\bullet, \phi_\alpha))(0, \mathrm{id}_{V^\bullet}, 0)\, s_\alpha = (-\mathrm{id}_{T(V^\bullet)}, 0, 0)\, s_\alpha = \alpha.$$

It remains to show that $M_\alpha^\bullet$ has finite pseudocompact $k[\varepsilon]$-tor dimension. Let N be an integer with $\mathrm{H}^j(V^\bullet) = 0$ for $j < N$. Since $M_\alpha^\bullet = C(T^{-1}(\alpha))^\bullet$ lies in the triangle

$$T^{-1}(V^\bullet) \xrightarrow{T^{-1}(\alpha)} V^\bullet \longrightarrow M_\alpha^\bullet \longrightarrow V^\bullet,$$

we obtain a long exact cohomology sequence

$$\cdots \to \mathrm{H}^{j-1}(V^\bullet) \longrightarrow \mathrm{H}^j(V^\bullet) \longrightarrow \mathrm{H}^j(M_\alpha^\bullet) \longrightarrow \mathrm{H}^j(V^\bullet) \to \cdots.$$

Hence $\mathrm{H}^j(M_\alpha^\bullet) = 0$ for all $j < N$. To show that $M_\alpha^\bullet$ has finite pseudocompact $k[\varepsilon]$-tor dimension, it is enough to show that the quotient $M_\alpha^N/\delta_M^{N-1}(M_\alpha^{N-1})$ is topologically free over $k[\varepsilon]$, because then we can truncate $M_\alpha^\bullet$ to obtain a quasi-isomorphic bounded complex of topologically free $k[\varepsilon]$-modules. Since M_α^N is abstractly free over $k[\varepsilon]$, using Baer's Criterion, it is enough to show that $\delta_M^{N-1}(M_\alpha^{N-1})$ is abstractly free over $k[\varepsilon]$. We have

$$B^N(M_\alpha^\bullet) = \delta_M^{N-1}(M_\alpha^{N-1}) = \big\{(\delta_V^{N-1}(a), \alpha^{N-1}(a) + \delta_V^{N-1}(b)) \mid a,b \in V^{N-1}\big\},$$

and we have a short exact sequence

$$(6.10) \quad \begin{cases} 0 \to K \to B^N(M_\alpha^\bullet) \xrightarrow{\eta} B^N(V^\bullet) \to 0, \quad \text{where} \\ (\delta_V^{N-1}(a), \alpha^{N-1}(a) + \delta_V^{N-1}(b)) \xmapsto{\eta} \delta_V^{N-1}(a). \end{cases}$$

To describe the kernel K, consider $a \in V^{N-1}$ with $\delta_V^{N-1}(a) = 0$. Since $V^\bullet$ is exact in dimensions less than N, we have that a has the form $a = \delta_V^{N-2}(a')$ for some $a' \in V^{N-2}$. Since $\alpha : V^\bullet \to T(V^\bullet)$ is in $C^-([[kG]])$, we have

$$\alpha^{N-1}(a) = \alpha^{N-1}\delta_V^{N-2}(a') = -\delta_V^{N-1}\alpha^{N-2}(a'),$$

and

$$(6.11) \quad \begin{aligned} K &= \left\{ (0, \alpha^{N-1}(a) + \delta_V^{N-1}(b)) \mid a, b \in V^{N-1}, \delta_V^{N-1}(a) = 0 \right\} \\ &= \left\{ (0, \delta_V^{N-1}(c)) \mid c \in V^{N-1} \right\} = \left(0, B^N(V^\bullet) \right). \end{aligned}$$

The action of ε sends $(\delta_V^{N-1}(a), \alpha^{N-1}(a) + \delta_V^{N-1}(b))$ to $(0, \delta_V^{N-1}(a))$. Suppose $\{d_\ell\}$ is a k-basis for $B^N(V^\bullet)$, i.e. $d_\ell = \delta_V^{N-1}(a_\ell)$ for some $a_\ell \in V^{N-1}$. We claim that then

$$(6.12) \qquad\qquad \left\{ (\delta_V^{N-1}(a_\ell), \alpha^{N-1}(a_\ell)) \right\}$$

is a $k[\varepsilon]$-basis for $B^N(M_\alpha^\bullet)$. This follows, since the short exact sequence (6.10) has a splitting over k, given by

$$\left(\delta_V^{N-1}(a_\ell), \alpha^{N-1}(a_\ell) \right) \xleftarrow{\eta'} \delta_V^{N-1}(a_\ell),$$

$$0 \to K \longrightarrow B^N(M_\alpha^\bullet) \xrightleftharpoons[\eta]{\eta'} B^N(V^\bullet) \to 0,$$

$$\left(\delta_V^{N-1}(a), \alpha^{N-1}(a) + \delta_V^{N-1}(b) \right) \xmapsto{\eta} \delta_V^{N-1}(a).$$

Further, by (6.11), $\{(0, d_\ell)\} = \{(0, \delta_V^{N-1}(a_\ell))\}$ is a k-basis for K. Since

$$\varepsilon\left(\delta_V^{N-1}(a_\ell), \alpha^{N-1}(a_\ell) \right) = \left(0, \delta_V^{N-1}(a_\ell) \right),$$

it follows that $\delta_M^{N-1}(M_\alpha^{N-1}) = B^N(M_\alpha^\bullet)$ is an abstractly free $k[\varepsilon]$-module on the basis given in (6.12). Therefore, the map h is surjective, which proves Claim 3.

Claim 4. — The map h is k-linear.

Proof of Claim 4. — Let $\alpha, \beta \in \mathrm{Hom}_{D^-([[kG]])}(V^\bullet, T(V^\bullet))$. As in the proof of Claim 3, we can take α and β to be represented by morphisms in $C^-([[kG]])$. Let $M_\alpha^\bullet$ and $M_\beta^\bullet$ be defined analogously to (6.9), and let $\lambda \in k$. Since Schlessinger's criterion (H2) is valid by Lemma 5.4, it follows from [25, Lemma 2.10] that t_F has a vector space structure with

---

scalar multiplication $\cdot_t$ and addition $+_t$. We first show that h preserves scalar multiplication, i.e. $\lambda \cdot_t M_\alpha^\bullet \cong M_{\lambda\alpha}^\bullet$ in $D^-([[k[\varepsilon]G]])$. The complex $\lambda \cdot_t M_\alpha^\bullet$ is defined to be the tensor product $k[\varepsilon] \otimes_{k[\varepsilon], \mu_\lambda} M_\alpha^\bullet$, where μ_λ is the W-algebra homomorphism $\mu_\lambda : k[\varepsilon] \to k[\varepsilon]$ defined by $\mu_\lambda(x \oplus \varepsilon y) = x \oplus \varepsilon\lambda y$. If $\lambda = 0$, we have

$$0 \cdot_t M_\alpha^\bullet = k[\varepsilon] \otimes_{k[\varepsilon], \mu_0} M_\alpha^\bullet = k[\varepsilon] \otimes_k (k \otimes_{k[\varepsilon]} M_\alpha^\bullet) = k[\varepsilon] \otimes_k V^\bullet.$$

The latter is the trivial lift of $V^\bullet$ over $k[\varepsilon]$, and this is isomorphic to $M_0^\bullet$. Let now $\lambda \neq 0$. Then

$$\lambda \cdot_t M_\alpha^\bullet = V^\bullet \oplus V^\bullet \quad \text{with } i\text{-th differential } \delta_{\lambda \cdot_t M_\alpha}^i = \begin{pmatrix} \delta_V^i & 0 \\ \alpha^i & \delta_V^i \end{pmatrix}$$

and with ε-action given by $\varepsilon(a, b) = \mu_\lambda(\varepsilon\lambda^{-1})(a, b) = (0, \lambda^{-1}a)$. We define $\sigma_\lambda : M_{\lambda\alpha}^\bullet \to \lambda \cdot_t M_\alpha^\bullet$ by $\sigma_\lambda^i(a, b) = (a, \lambda^{-1}b)$ for all $a, b \in V^i$. Then σ_λ commutes with the differentials, since

$$\begin{pmatrix} \delta_V^i & 0 \\ \alpha^i & \delta_V^i \end{pmatrix} \sigma_\lambda^i = \begin{pmatrix} \delta_V^i & 0 \\ \alpha^i & \lambda^{-1}\delta_V^i \end{pmatrix} = \sigma_\lambda^{i+1} \begin{pmatrix} \delta_V^i & 0 \\ \lambda\alpha^i & \delta_V^i \end{pmatrix}.$$

Hence σ_λ is a morphism in $C^-([[kG]])$. Moreover, $\sigma_\lambda^i(\varepsilon(a, b)) = \sigma_\lambda^i(0, a) = (0, \lambda^{-1}a) = \varepsilon(a, \lambda^{-1}b) = \varepsilon\sigma_\lambda^i(a, b)$. Since for all i, σ_λ^i is continuous and bijective, it follows that $\lambda \cdot_t M_\alpha^\bullet \cong M_{\lambda\alpha}^\bullet$ in $C^-([[k[\varepsilon]G]])$. Since σ_λ obviously identifies $M_{\lambda\alpha}^\bullet / \varepsilon M_{\lambda\alpha}^\bullet = V^\bullet = \lambda \cdot_t M_\alpha^\bullet / \varepsilon(\lambda \cdot_t M_\alpha^\bullet)$, it follows that h preserves scalar multiplication.

We now show that h preserves addition, i.e. $M_\alpha^\bullet +_t M_\beta^\bullet \cong M_{\alpha+\beta}^\bullet$ in $D^-([[k[\varepsilon]G]])$. If we denote $M_\alpha^\bullet +_t M_\beta^\bullet$ by $M_+^\bullet$, then $M_+^\bullet$ is defined to be the tensor product $k[\varepsilon] \otimes_{k[\varepsilon] \times_k k[\varepsilon], +} (M_\alpha^\bullet \times_{V^\bullet} M_\beta^\bullet)$, where $+$ is the surjective k-algebra homomorphism

$$+ : k[\varepsilon] \times_k k[\varepsilon] \longrightarrow k[\varepsilon], \quad (x \oplus \varepsilon y_1, x \oplus \varepsilon y_2) \longmapsto x \oplus \varepsilon(y_1 + y_2).$$

If we denote $M_\alpha^\bullet \times_{V^\bullet} M_\beta^\bullet$ by $Z^\bullet$, then we have for all i,

$$Z^i = \left\{ (a_1, b_1, a_1, b_2) \mid a_1, b_1, b_2 \in V^i \right\},$$

$$\delta_Z^i(a_1, b_1, a_1, b_2) = \left(\delta_V^i(a_1), \alpha^i(a_1) + \delta_V^i(b_1), \delta_V^i(a_1), \beta^i(a_1) + \delta_V^i(b_2) \right), \text{ and}$$

$$(x \oplus \varepsilon y_1, x \oplus \varepsilon y_2)(a_1, b_1, a_1, b_2) = (xa_1, y_1 a_1 + xb_1, xa_1, y_2 a_1 + xb_2).$$

Since $+$ is surjective, it follows that

$$M_+^\bullet = k[\varepsilon] \otimes_{k[\varepsilon] \times_k k[\varepsilon], +} Z^\bullet = Z^\bullet / \mathrm{Ker}(+)Z^\bullet.$$

We have $\mathrm{Ker}(+) = \{(\varepsilon y, -\varepsilon y)\}$, and

$$(\varepsilon y, -\varepsilon y)(a_1, b_1, a_1, b_2) = (0, ya_1, 0, -ya_1)$$

for all $a_1, b_1, b_2 \in V^i$. If we use the notation

$$\overline{(a_1, b_1, a_1, b_2)} = (a_1, b_1, a_1, b_2) + \mathrm{Ker}(+)Z^i,$$

it follows for all i that

$$M_+^i = \left\{ \overline{(a_1, 0, a_1, b_2)} \mid a_1, b_2 \in V^i \right\}$$

and

$$\delta_{M_+}^i \left(\overline{(a_1, 0, a_1, b_2)} \right) = \overline{(\delta_V^i(a_1), \alpha^i(a_1), \delta_V^i(a_1), \beta^i(a_1) + \delta_V^i(b_2))}$$
$$= \overline{(\delta_V^i(a_1), 0, \delta_V^i(a_1), (\alpha^i + \beta^i)(a_1) + \delta_V^i(b_2))}.$$

We define $\tau : M_+^\bullet \to M_{\alpha+\beta}^\bullet$ by $\tau^i(\overline{(a_1, 0, a_1, b_2)}) = (a_1, b_2)$. Then τ is well-defined and commutes with the differentials, since

$$\tau^{i+1}\left(\delta_{M_+}^i \left(\overline{(a_1, 0, a_1, b_2)} \right) \right)$$
$$= \tau^{i+1}\left(\overline{(\delta_V^i(a_1), 0, \delta_V^i(a_1), (\alpha^i + \beta^i)(a_1) + \delta_V^i(b_2))} \right)$$
$$= \left(\delta_V^i(a_1), (\alpha^i + \beta^i)(a_1) + \delta_V^i(b_2) \right)$$
$$= \delta_{M_{\alpha+\beta}}^i(a_1, b_2) = \delta_{M_{\alpha+\beta}}^i\left(\tau^i(\overline{(a_1, 0, a_1, b_2)}) \right).$$

Hence τ is a morphism in $C^-([[kG]])$. Moreover,

$$\tau^i(\varepsilon \overline{(a_1, 0, a_1, b_2)}) = \tau^i((0, \varepsilon)(a_1, 0, a_1, b_2)) = \tau^i(\overline{(0, 0, 0, a_1)})$$
$$= (0, a_1) = \varepsilon(a_1, b_2) = \varepsilon \tau^i(\overline{(a_1, 0, a_1, b_2)}).$$

Since for all i, τ^i is continuous and bijective, it follows that

$$M_\alpha^\bullet +_t M_\beta^\bullet \cong M_{\alpha+\beta}^\bullet$$

in $C^-([[k[\varepsilon]G]])$. Since τ obviously identifies

$$M_{\alpha+\beta}^\bullet / \varepsilon M_{\alpha+\beta}^\bullet = V^\bullet = (M_\alpha^\bullet +_t M_\beta^\bullet)/\varepsilon(M_\alpha^\bullet +_t M_\beta^\bullet),$$

it follows that h preserves addition. This proves Claim 4, and completes the proof of Lemma 6.1.   □

Remark 6.2. — As seen in the proof of Lemma 6.1, an element

$$\alpha \in \mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet, V^\bullet) = \mathrm{Hom}_{D^-([[kG]])}\left(V^\bullet, T(V^\bullet)\right)$$

defines a quasi-lift, and hence a deformation, of $V^\bullet$ over $k[\varepsilon]$ as follows. By Proposition 2.12, we may assume that $V^\bullet$ is a bounded above complex of topologically free pseudocompact $[[kG]]$-modules. Hence $\alpha : V^\bullet \to T(V^\bullet)$ can be represented by a morphism in $C^-([[kG]])$. We can define an ε-action on the mapping cone $C(T^{-1}(\alpha))^\bullet = V^\bullet \oplus V^\bullet$ by $\varepsilon(a, b) = (0, a)$ for all $(a, b) \in C(T^{-1}(\alpha))^i$. Then the complex $M^\bullet = C(T^{-1}(\alpha))^\bullet$ defines a quasi-lift $(M^\bullet, \phi)$ of $V^\bullet$ over $k[\varepsilon]$.

LEMMA 6.3. — *The composition of the natural map $t_{F^{\mathrm{fl}}} \to t_F$ and the isomorphism h from Lemma* 6.1 *induces an isomorphism between $t_{F^{\mathrm{fl}}}$ and the kernel of the natural map*

$$(6.13) \qquad \mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet, V^\bullet) \longrightarrow \mathrm{Ext}^1_{D^-([[k]])}(V^\bullet, V^\bullet)$$

given by forgetting the G-action.

Proof. — By Lemma 6.1, $t_{F^{\mathrm{fl}}}$ is isomorphic to the subspace of $\mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet, V^\bullet)$ consisting of those elements which define proflat deformations of $V^\bullet$ over $k[\varepsilon]$. Let $(M^\bullet, \phi)$ be a quasi-lift of $V^\bullet$ over $k[\varepsilon]$. Then $(M^\bullet, \phi)$ is a proflat quasi-lift if the cohomology groups of $M^\bullet$ are topologically free pseudocompact, and hence abstractly free, $k[\varepsilon]$-modules. In this case $M^\bullet$ is isomorphic in $D^-([[k[\varepsilon]]])$ to a bounded complex with trivial boundary maps whose term in dimension n is a free $k[\varepsilon]$-module of rank $\dim_k \mathrm{H}^n(V^\bullet)$ for all integers n. Therefore, all proflat quasi-lifts $M^\bullet$ of $V^\bullet$ over $k[\varepsilon]$ are isomorphic in $D^-([[k[\varepsilon]]])$, when we forget the G-action. This means that the tangent space $t_{F^{\mathrm{fl}}}$ is mapped to $\{0\}$ under the forgetful map (6.13). Suppose now that $\alpha \in \mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet, V^\bullet)$ is mapped to the zero map under (6.13). Then by the proof of Lemma 6.1, the quasi-lift $M^\bullet$ of $V^\bullet$ over $k[\varepsilon]$ corresponding to α is isomorphic to the trivial lift $X^\bullet = k[\varepsilon] \otimes_k V^\bullet$ in $D^-([[k[\varepsilon]]])$. Since $V^\bullet$ is completely split in $D^-([[k]])$, it follows that the cohomology groups of $X^\bullet$ are abstractly free, and hence topologically free, over $k[\varepsilon]$. Thus $M^\bullet$ defines a proflat quasi-lift of $V^\bullet$ over $k[\varepsilon]$. This completes the proof of Lemma 6.3.   □

PROPOSITION 6.4. — *Suppose G has finite pseudocompact cohomology. Then Schlessinger's criterion* (H3) *is satisfied, i.e. the k-dimension of the tangent space $t_{F_\mathcal{D}}$ is finite.*

Proof. — By Lemmas 6.1 and 6.3, it is enough to find an upper bound for the k-dimension of $\mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet, V^\bullet)$. By truncating and shifting, we can assume that $V^\bullet$ has the form

$$V^\bullet : \cdots \to 0 \to V^{-n} \longrightarrow V^{-n+1} \longrightarrow \cdots \longrightarrow V^0 \to 0 \to \cdots.$$

Claim. — Suppose n is a non-negative integer, and $L_1^\bullet$ (resp. $L_2^\bullet$) is a complex of pseudocompact $[[kG]]$-modules whose terms are concentrated between dimensions $-n_1$ and $-n_1 + n$ (resp. between $-n_2$ and $-n_2 + n$), for integers n_1 and n_2. Then for all integers j, $\mathrm{Ext}^j_{D^-([[kG]])}(L_1^\bullet, L_2^\bullet)$ has finite k-dimension, if all cohomology groups of $L_1^\bullet$ and of $L_2^\bullet$ have finite k-dimension.

Once we have proved this claim, Proposition 6.4 follows by setting $L_1^\bullet = V^\bullet = L_2^\bullet$ and $j = 1$.

Proof of Claim. — We prove the claim by induction on n. If $n = 0$, then $L_1^\bullet$ (resp. $L_2^\bullet$) is a module in dimension $-n_1$ (resp. $-n_2$). Hence

$$\mathrm{Ext}^j_{D^-([[kG]])}(L_1^\bullet, L_2^\bullet) = \mathrm{Ext}^j_{[[kG]]}\big(T^{n_1}\big(\mathrm{H}^{-n_1}(L_1^\bullet)\big), T^{n_2}\big(\mathrm{H}^{-n_2}(L_2^\bullet)\big)\big)$$
$$= \mathrm{Ext}^{j+n_2-n_1}_{[[kG]]}(L_1^{-n_1}, L_2^{-n_2})$$

which has finite k-dimension, since G has finite pseudocompact cohomology. Suppose now that $n > 0$. For $i = 1, 2$, define $Z^{-n_i}(L_i^\bullet) = \mathrm{Ker}(\delta_{L_i}^{-n_i})$. Then the diagram

$$
\begin{array}{ccccccccc}
T^{n_i}\big(\mathrm{H}^{-n_i}(L_i^\bullet)\big) : & 0 \to & Z^{-n_i}(L_i^\bullet) & \longrightarrow & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
L_i^\bullet : & 0 \to & L_i^{-n_i} & \longrightarrow & L_i^{-n_i+1} & \to \cdots \to & L_i^{-n_i+n} & \to 0 \\
& & \downarrow & & \| & & \| & & \\
\widetilde{L}_i^\bullet : & 0 \to & L_i^{-n_i}/Z^{-n_i}(L_i^\bullet) & \to & L_i^{-n_i+1} & \to \cdots \to & L_i^{-n_i+n} & \to 0
\end{array}
$$

induces a short exact sequence of complexes

$$(6.14) \qquad 0 \to T^{-n_i}\big(\mathrm{H}^{-n_i}(L_i^\bullet)\big) \longrightarrow L_i^\bullet \longrightarrow \widetilde{L}_i^\bullet \to 0.$$

The complex $\widetilde{L}_i^\bullet$ is quasi-isomorphic to the complex

$$L_{i,1}^\bullet : \cdots \to 0 \to L_i^{-n_i+1}/\delta_{L_i}^{-n_i}(L_i^{-n_i}) \longrightarrow L_i^{-n_i+2}$$
$$\to \cdots \to L_i^{-n_i+n} \to 0 \to \cdots$$

with terms concentrated between dimensions $-n_i + 1$ and $-n_i + n$. By [17,

Prop. I.6.1], we have long exact Ext sequences

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{Ext}^j\big(L_{1,1}^\bullet, T^{n_2}\big(\mathrm{H}^{-n_2}(L_2^\bullet)\big)\big) & & \mathrm{Ext}^j\big(T^{n_1}\big(\mathrm{H}^{-n_1}(L_1^\bullet)\big), T^{n_2}\big(\mathrm{H}^{-n_2}(L_2^\bullet)\big)\big) \\
\downarrow & & \downarrow \\
\cdots \to \mathrm{Ext}^j(L_{1,1}^\bullet, L_2^\bullet) \longrightarrow \mathrm{Ext}^j(L_1^\bullet, L_2^\bullet) \longrightarrow \mathrm{Ext}^j\big(T^{n_1}\big(\mathrm{H}^{-n_1}(L_1^\bullet)\big), L_2^\bullet\big) \to \cdots \\
\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\mathrm{Ext}^j(L_{1,1}^\bullet, L_{2,1}^\bullet) \qquad\qquad\qquad\qquad \mathrm{Ext}^j\big(T^{n_1}\big(\mathrm{H}^{-n_1}(L_1^\bullet)\big), L_{2,1}^\bullet\big) \\
\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots
\end{array}
$$

where $\mathrm{Ext}^j = \mathrm{Ext}^j_{D^-([[kG]])}$ for all integers j. Since each complex involved in the first and third row has non-zero terms only between $n-1$ consecutive dimensions, the claim follows by induction. Therefore, Proposition 6.4 is proved. $\qquad\square$

COROLLARY 6.5. — *Suppose G has finite pseudocompact cohomology. Then $F_{\mathcal{D}}$ has a pro-representable hull. If $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet) = k$, then $F_{\mathcal{D}}$ is pro-representable.*

Proof. — This follows from [25, Thm. 2.11], using Propositions 5.1 and 6.4. $\qquad\square$

7. Continuity.

In this section we finish the proof of Theorem 2.14, using a continuity argument.

DEFINITION 7.1. — A functor $D : \widehat{\mathcal{C}} \to \mathrm{Sets}$ is said to be *continuous*, if for all objects R in $\widehat{\mathcal{C}}$ with maximal ideal m_R we have

$$D(R) = \varprojlim_i D(R/m_R^i).$$

By [25, §2], the proof of Theorem 2.14 follows from Corollary 2.14 and the next result.

PROPOSITION 7.2. — *The functor $\widehat{F}_{\mathcal{D}} : \widehat{\mathcal{C}} \to \mathrm{Sets}$ is continuous.*

Proof. — By Proposition 2.12 and Corollary 3.6 (i) we may assume, without loss of generality, that there is a closed normal subgroup Δ of finite index in G such that $V^\bullet$ is a bounded above complex of abstractly free finitely generated $[k(G/\Delta)]$-modules. Let R be an object in $\widehat{\mathcal{C}}$ with maximal ideal m_R. We first show that the natural map

$$(7.1) \qquad \Gamma_\mathcal{D} : \widehat{F}_\mathcal{D}(R) \longrightarrow \varprojlim_i \widehat{F}_\mathcal{D}(R/m_R^i)$$

defined by

$$\Gamma_\mathcal{D}((M^\bullet, \phi)) = \left\{ \left((R/m_R^i)\widehat{\otimes}_R^{\boldsymbol{L}} M^\bullet, (R/m_R^i)\widehat{\otimes}_R^{\boldsymbol{L}} \phi \right) \right\}_{i=1}^\infty$$

is surjective.

Suppose we have a sequence $\{(M_i^\bullet, \phi_i)\}_{i=1}^\infty$ with $(M_i^\bullet, \phi_i)$ in $\widehat{F}_\mathcal{D}(R/m_R^i)$ $= F_\mathcal{D}(R/m_R^i)$ for all i such that there is an isomorphism

$$\alpha_i : (R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}}^{\boldsymbol{L}} M_{i+1}^\bullet \longrightarrow M_i^\bullet$$

in $D^-([[(R/m_R^i)G]])$ with $\phi_i (k\widehat{\otimes}_{R/m_R^i}^{\boldsymbol{L}} \alpha_i) = \phi_{i+1}$. We need to construct an element $(M^\bullet, \phi) \in \widehat{F}_\mathcal{D}(R)$ such that, for all i, there is an isomorphism

$$\beta_i : (R/m_R^i)\widehat{\otimes}_R^{\boldsymbol{L}} M^\bullet \longrightarrow M_i^\bullet$$

in $D^-([[(R/m_R^i)G]])$ with

$$\alpha_i\big((R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}}^{\boldsymbol{L}} \beta_{i+1}\big) = \beta_i \quad \text{and} \quad \phi_i(k\widehat{\otimes}_{R/m_R^i}^{\boldsymbol{L}} \beta_i) = \phi.$$

We construct $M^\bullet$ inductively.

By Corollary 3.11 (i), there exists for each i a closed normal subgroup Δ_i of finite index in G with $\Delta_i \subseteq \Delta$ having the following properties: We can replace $M_i^\bullet$ by a bounded above complex $N_i^\bullet$ of abstractly free finitely generated $[(R/m_R^i)(G/\Delta_i)]$-modules, and we can replace ϕ_i by a quasi-isomorphism

$$\psi_i : k\widehat{\otimes}_{R/m_R^i} N_i^\bullet \longrightarrow \mathrm{Inf}_{G/\Delta}^{G/\Delta_i} V^\bullet$$

in $C^-([k(G/\Delta_i)])$. Suppose $\gamma_i : \mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet \to M_i^\bullet$ an isomorphism in $D^-([[(R/m_R^i)G]])$ associated to these replacements. Then the diagram

$$(7.2) \qquad (R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}} \mathrm{Inf}_{G/\Delta_{i+1}}^G N_{i+1}^\bullet$$

$$\xrightarrow{\;(R/m_R^i)\widehat{\otimes}^{\boldsymbol{L}}\gamma_{i+1}\;} (R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}}^{\boldsymbol{L}} M_{i+1}^\bullet$$

$$\downarrow{\alpha_i}$$

$$M_i^\bullet$$

$$\downarrow{\gamma_i^{-1}}$$

$$\mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet$$

defines an isomorphism

$$\widetilde{\alpha}_i : (R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}} \mathrm{Inf}_{G/\Delta_{i+1}}^G N_{i+1}^\bullet \longrightarrow \mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet$$

in $D^-([[(R/m_R^i)G]])$ such that $\mathrm{Inf}_{G/\Delta_i}^G(\psi_i)(k\widehat{\otimes}_{R/m_R^i}\widetilde{\alpha}_i) = \mathrm{Inf}_{G/\Delta_{i+1}}^G(\psi_{i+1})$. By Corollary 3.11 (ii), we can shrink Δ_{i+1} to be able to assume that $\widetilde{\alpha}_i$ is represented by a quasi-isomorphism

$$\widetilde{\alpha}_i : (R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}} N_{i+1}^\bullet \longrightarrow \mathrm{Inf}_{G/\Delta_i}^{G/\Delta_{i+1}} N_i^\bullet$$

in $C^-([(R/m_R^i)(G/\Delta_{i+1})])$. By Remark 5.2, we can add to $N_{i+1}^\bullet$ a suitable acyclic bounded above complex of abstractly free finitely generated $[(R/m_R^{i+1})(G/\Delta_{i+1})]$-modules to be able to assume that $\widetilde{\alpha}_i$ is surjective on terms. Continuing this process inductively, we obtain an inverse system

$$\left\{ (\mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet, \mathrm{Inf}_{G/\Delta_i}^G(\psi_i)) \right\}_{i=1}^\infty,$$

where $(\mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet, \mathrm{Inf}_{G/\Delta_i}^G \psi_i) \in F_\mathcal{D}(R/m_R^i)$. Further, $N_i^\bullet$ is a complex of abstractly free finitely generated $[(R/m_R^i)(G/\Delta_i)]$-modules such that in the diagram

$$(7.3) \qquad \begin{array}{c} N_{i+1}^\bullet \\ \downarrow \\ (R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}} N_{i+1}^\bullet \xrightarrow{\;\widetilde{\alpha}_i\;} \mathrm{Inf}_{G/\Delta_i}^{G/\Delta_{i+1}} N_i^\bullet \end{array}$$

all arrows are morphisms in $C^-((R/m_R^{i+1})(G/\Delta_{i+1}))$ which are surjective on terms. We now set $M^\bullet = \varprojlim_i \mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet$ and $\phi = \varprojlim_i \mathrm{Inf}_{G/\Delta_i}^G(\psi_i)$. We claim that $(M^\bullet, \phi) \in \widehat{F}_\mathcal{D}(R)$.

We first show that the terms of $M^\bullet$ are topologically free pseudocompact R-modules. This follows, since the terms of $\mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet$ are abstractly free finitely generated (R/m_R^i)-modules for all i. Hence the j-th term of $\mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet$ has the form $\mathrm{Inf}_{G/\Delta_i}^G N_i^j = [(R/m_R^i)X_i^j]$ for some finite space X_i^j endowed with the discrete topology, using the notation of Remark 2.3. Since $\mathrm{Inf}_{G/\Delta_{i+1}}^G N_{i+1}^j$ surjects onto $\mathrm{Inf}_{G/\Delta_i}^G N_i^j$ by (7.3), the finite space X_{i+1}^j surjects onto the finite space X_i^j. Hence $\{X_i^j\}_i$ forms an inverse system, and we define $X^j = \varprojlim_i X_i^j$. By Remark 2.3,

$$M^j = \varprojlim_i \mathrm{Inf}_{G/\Delta_i}^G N_i^j = \varprojlim_i [(R/m_R^i)X_i^j] = [[RX^j]]$$

is a topologically free pseudocompact R-module on X^j.

We now show that $M^\bullet$ has finite pseudocompact R-tor dimension. Suppose S is an arbitrary pseudocompact R-module. By assumption, $M_i^\bullet$, and thus $N_i^\bullet$, has finite pseudocompact (R/m_R^i)-tor dimension for all i. By Lemma 3.1 and Lemma 3.8, it follows that there exists an integer n so that for all i

$$\mathrm{H}^j\big((S/m_R^i S)\widehat{\otimes}_{R/m_R^i} N_i^\bullet\big) = 0 \quad \text{for all } j < n.$$

For all $N_i^\bullet$, we have for all j short exact sequences of coboundaries and cocycles

$$0 \to B^j(N_i^\bullet) \longrightarrow Z^j(N_i^\bullet) \longrightarrow \mathrm{H}^j(N_i^\bullet) \to 0$$

where $B^j(N_i^\bullet) = B^j(\mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet)$, $Z^j(N_i^\bullet) = Z^j(\mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet)$ and $\mathrm{H}^j(N_i^\bullet) = \mathrm{H}^j(\mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet)$ as pseudocompact R-modules. By Remark 2.2 (i),

$$0 \to \varprojlim_i B^j(N_i^\bullet) \longrightarrow \varprojlim_i Z^j(N_i^\bullet) \longrightarrow \varprojlim_i \mathrm{H}^j(N_i^\bullet) \to 0$$

stays exact, and

$$\varprojlim_i B^j(N_i^\bullet) = B^j(\varprojlim_i N_i^\bullet) \quad \text{and} \quad \varprojlim_i Z^j(N_i^\bullet) = Z^j(\varprojlim_i N_i^\bullet).$$

Since the j-th differential of $M^\bullet$ is $\delta_M^j = \varprojlim_i \mathrm{Inf}_{G/\Delta_i}^G \delta_{N_i}^j$, it follows that

$$(7.4) \qquad \mathrm{H}^j(M^\bullet) = \mathrm{H}^j(\varprojlim_i N_i^\bullet) = \varprojlim_i \mathrm{H}^j(N_i^\bullet)$$

as pseudocompact R-modules. Similarly we get for all $j < n$,

$$\mathrm{H}^j(S \widehat{\otimes}_R M^\bullet) = \mathrm{H}^j\big(\varprojlim_i\big((S/m_R^i S)\widehat{\otimes}_{R/m_R^i} N_i^\bullet\big)\big)$$
$$= \varprojlim_i \mathrm{H}^j\big((S/m_R^i S)\widehat{\otimes}_{R/m_R^i} N_i^\bullet\big) = 0,$$

which implies that $M^\bullet$ has finite pseudocompact R-tor dimension.

In case $\widehat{F}_\mathcal{D} = \widehat{F}^{\mathrm{fl}}$, we have for all j and for all i that $\mathrm{H}^j(N_i^\bullet) = \mathrm{H}^j(M_i^\bullet)$ is an abstractly free (R/m_R^i)-module of rank $d_j = \dim_k \mathrm{H}^j(V^\bullet)$ by Lemma 2.11. As in (2.1), we obtain inductively that the terms of $N_i^\bullet$ split completely as

$$N_i^j = \delta_{N_i}^{j-1}(C_i^{j-1}) \oplus Y_i^j \oplus C_i^j$$

where $\delta_{N_i}^{j-1}(C_i^{j-1}) = \mathrm{Image}(\delta_{N_i}^{j-1})$ and $\delta_{N_i}^{j-1}(C_i^{j-1}) \oplus Y_i^j = \mathrm{Ker}(\delta_{N_i}^j)$ as (R/m_R^i)-modules. We get a commutative diagram

$$
\begin{array}{ccccccc}
N_{i+1}^\bullet: & \cdots \longrightarrow & N_{i+1}^{j-1} & \xrightarrow{\ \delta_{N_{i+1}}^{j-1}\ } & N_{i+1}^j & \longrightarrow \cdots \\
& & \Big\downarrow{\tau_{i+1}^{j-1}} & & \Big\downarrow{\tau_{i+1}^j} & \\
N_i^\bullet: & \cdots \longrightarrow & N_i^{j-1} & \xrightarrow{\ \delta_{N_i}^{j-1}\ } & N_i^j & \longrightarrow \cdots
\end{array}
$$

where τ_{i+1} is the morphism given by the composition of the two morphisms in (7.3). Hence τ_{i+1}^j is a surjective (R/m_R^i)-module homomorphism for all j. Using that $N_{i+1}^\bullet$ and $N_i^\bullet$ are bounded above complexes, it follows inductively that $\tau_{i+1}^j(Y_{i+1}^j) = Y_i^j$ for all j. Since $Y_i^j = \mathrm{H}^j(N_i^\bullet)$ is an abstractly free (R/m_R^i)-module of rank d_j, there exists a finite space Z_i^j of order d_j such that $\mathrm{H}^j(N_i^\bullet) = [(R/m_R^i)Z_i^j]$, using the notation of Remark 2.3. Because τ_{i+1}^j maps $\mathrm{H}^j(N_{i+1}^\bullet)$ surjectively onto $\mathrm{H}^j(N_i^\bullet)$, the finite space Z_{i+1}^j surjects onto the finite space Z_i^j. Since these two finite spaces both have order d_j, this surjection is in fact a bijection. Hence $\{Z_i^j\}_i$ forms an inverse system, and $Z^j = \varprojlim_i Z_i^j$ is a finite space of order d_j. By equation (7.4), it follows that $\mathrm{H}^j(M^\bullet) = [RZ^j]$ is an abstractly free R-module of rank d_j.

Using the natural isomorphisms

$$\widetilde{\beta}_i : (R/m_R^i)\widehat{\otimes}_R M^\bullet = (R/m_R^i)\widehat{\otimes}_R \varprojlim_i \mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet \longrightarrow \mathrm{Inf}_{G/\Delta_i}^G N_i^\bullet$$

in $C^-([[RG]])$, it follows that $\mathrm{Inf}_{G/\Delta_i}^G(\widetilde{\alpha}_i)((R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}}\widetilde{\beta}_{i+1}) = \widetilde{\beta}_i$, where $\widetilde{\alpha}_i$ is as defined in (7.2), and $\mathrm{Inf}_{G/\Delta_i}^G(\psi_i)(k\widehat{\otimes}_{R/m_R^i}\widetilde{\beta}_i) = \phi$. Hence, if $\beta_i = \gamma_i \widetilde{\beta}_i$, we have $\alpha_i((R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}}^{\boldsymbol{L}}\beta_{i+1}) = \beta_i$ and $\phi_i(k\widehat{\otimes}_{R/m_R^i}^{\boldsymbol{L}}\beta_i) = \phi$. It follows that $(M^\bullet, \phi) \in \widehat{F}_\mathcal{D}(R)$, and the map $\Gamma_\mathcal{D}$ in (7.1) is surjective.

We now show that $\Gamma_\mathcal{D}$ is injective. Since $\widehat{F}^{\mathrm{fl}}$ is a subfunctor of $\widehat{F}$ by Proposition 2.12, it is enough to show that $\Gamma_\mathcal{D}$ is injective in case $\widehat{F}_\mathcal{D} = \widehat{F}$. We abbreviate $\Gamma_\mathcal{D}$ by Γ in this case. Let $\{(M_i^\bullet, \phi_i)\}_{i=1}^\infty \in \varprojlim_i \widehat{F}(R/m_R^i)$, and let, for all i, $\alpha_i : (R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}}^{\boldsymbol{L}} M_{i+1}^\bullet \to M_i^\bullet$ be an isomorphism in $D^-([[(R/m_R^i)G]])$ with $\phi_i(k\widehat{\otimes}_{R/m_R^i}^{\boldsymbol{L}}\alpha_i) = \phi_{i+1}$. Suppose $(M^\bullet, \phi), (\widetilde{M}^\bullet, \widetilde{\phi})$ in $\widehat{F}(R)$ are such that $\Gamma((M^\bullet, \phi)) = \Gamma((\widetilde{M}^\bullet, \widetilde{\phi}))$, i.e. for all i, there are isomorphisms ξ_i (resp. $\widetilde{\xi}_i$) in $D^-([[(R/m_R^i)G]])$

$$
\begin{array}{ccc}
(R/m_R^i)\widehat{\otimes}_R^{\boldsymbol{L}} M^\bullet & & (R/m_R^i)\widehat{\otimes}_R^{\boldsymbol{L}} \widetilde{M}^\bullet \\
& \searrow{\scriptstyle \xi_i} \quad \swarrow{\scriptstyle \widetilde{\xi}_i} & \\
& M_i^\bullet &
\end{array}
$$

with $\alpha_i((R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}}^{\boldsymbol{L}}\xi_{i+1}) = \xi_i$ and $\phi_i(k\widehat{\otimes}_{R/m_R^i}^{\boldsymbol{L}}\xi_i) = \phi$ (respectively $\alpha_i((R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}}^{\boldsymbol{L}}\widetilde{\xi}_{i+1}) = \widetilde{\xi}_i$ and $\phi_i(k\widehat{\otimes}_{R/m_R^i}^{\boldsymbol{L}}\widetilde{\xi}_i) = \widetilde{\phi}$). By Lemma 4.2,

we can assume that the terms of $M^\bullet$ and $\widetilde{M}^\bullet$ are topologically free pseudocompact $[[RG]]$-modules. By Lemma 14.1, it follows that ξ_i (resp. $\widetilde{\xi}_i$) is an isomorphism in $K^-([[(R/m_R^i)G]])$ for all i, and thus

$$(7.5) \qquad f_i = \widetilde{\xi}_i^{-1}\xi_i : (R/m_R^i) \widehat{\otimes}_R M^\bullet \longrightarrow (R/m_R^i) \widehat{\otimes}_R \widetilde{M}^\bullet$$

is represented by a quasi-isomorphism f_i in $C^-([[(R/m_R^i)G]])$ for all i. By Remark 5.2, we can add to $M^\bullet$ a suitable acyclic bounded above complex of topologically free pseudocompact $[[RG]]$-modules to be able to assume that all f_i are surjective on terms. Define $h_1 = f_1$. By induction, assume that we have constructed a quasi-isomorphism

$$h_i : (R/m_R^i) \widehat{\otimes}_R M^\bullet \longrightarrow (R/m_R^i) \widehat{\otimes}_R \widetilde{M}^\bullet$$

in $C^-([[(R/m_R^i)G]])$ which is surjective on terms such that h_i is homotopic to f_i and such that $(R/m_R^j)\widehat{\otimes}_{R/m_R^i} h_i = h_j$ for all $j < i$. Let now $h_{i+1} = f_{i+1}$. Then $(R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}} h_{i+1}$ is homotopic to f_i, and thus to h_i. By Lemma 14.3, we can lift the homotopy so as to be able to assume that $(R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}} h_{i+1} = h_i$ in $C^-([[(R/m_R^i)G]])$. Since h_i is surjective on terms and since for all n $(R/m_R^{i+1}) \widehat{\otimes}_R M^n$ is a topologically free pseudocompact, and thus abstractly free, (R/m_R^{i+1})-module, it follows that h_{i+1} is also surjective on terms. Hence we obtain a sequence of quasi-isomorphisms $h_i : (R/m_R^i) \widehat{\otimes}_R M^\bullet \to (R/m_R^i) \widehat{\otimes}_R \widetilde{M}^\bullet$ in $C^-([[(R/m_R^i)G]])$ which are surjective on terms such that $(R/m_R^i)\widehat{\otimes}_{R/m_R^{i+1}} h_{i+1} = h_i$ in $C^-([[(R/m_R^i)G]])$. It follows that

$$\varprojlim_i h_i : M^\bullet = \varprojlim_i \big((R/m_R^i) \widehat{\otimes}_R M^\bullet \big) \longrightarrow \varprojlim_i \big((R/m_R^i) \widehat{\otimes}_R \widetilde{M}^\bullet \big) = \widetilde{M}^\bullet$$

is an isomorphism in $K^-([[RG]])$ with $\widetilde{\phi} \varprojlim_i h_i = \phi$. This completes the proof of Proposition 7.2. $\qquad \square$

8. Compatibility with finite extensions of scalars.

In this section we will assume the hypotheses of Theorem 2.14. Let k' be a finite field extension of k, let W' be a complete local commutative Noetherian ring with residue field k' which is faithfully flat over W, and define $V'^\bullet = k' \widehat{\otimes}_k^L V^\bullet$. Then $V'^\bullet$ is an object in $D^-([[k'G]])$ to which Theorem 2.14 applies when k is replaced by k'. Let $\widehat{\mathcal{C}}'$ (resp. $\mathcal{C}'$) be defined the same way as the category $\widehat{\mathcal{C}}$ (resp. $\mathcal{C}$) when k is replaced by k', and W is

replaced by W'. Define the deformation functors $\widehat{F}'_\mathcal{D}$ (resp. $F'_\mathcal{D}$) analogously to the functors $\widehat{F}_\mathcal{D}$ (resp. $F_\mathcal{D}$) when $V^\bullet$ is replaced by $V'^\bullet$, k is replaced by k' and $\widehat{\mathcal{C}}$ (resp. $\mathcal{C}$) is replaced by $\widehat{\mathcal{C}}'$ (resp. $\mathcal{C}'$). We thus have versal (proflat) deformation rings $R_\mathcal{D}(G, V^\bullet)$ and $R_\mathcal{D}(G, V'^\bullet)$, which are algebras over W and W', respectively. Let $U_\mathcal{D}(G, V^\bullet)$ be a versal (proflat) deformation of $V^\bullet$.

THEOREM 8.1. — *The complex* $W' \widehat{\otimes}_W^L U_\mathcal{D}(G, V^\bullet)$ *is a versal (proflat) deformation of* $V'^\bullet$, *and there is an isomorphism of* W'-*algebras* $W' \widehat{\otimes}_W R_\mathcal{D}(G, V^\bullet) \to R_\mathcal{D}(G, V'^\bullet)$.

Define $\widehat{X} : \widehat{\mathcal{C}}' \to \widehat{\mathcal{C}}$ to be the functor which sends an object $B \in \mathrm{Ob}(\widehat{\mathcal{C}}')$ to the subring $\widehat{X}(B)$ of B of all elements whose reduction modulo the maximal ideal m_B lie in k. Define $\widehat{X}$ on morphisms by restriction. (Note that in defining $\widehat{X}$, we have used the fact that k' is a finite extension of k. If ℓ is a field extension of k, the subring $k + \ell\varepsilon$ of the dual numbers $\ell[\varepsilon]$ over ℓ is Noetherian if and only if ℓ is a finite extension of k.) Let $\widehat{Y} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}'$ be the functor which sends an object A to $W' \widehat{\otimes}_W A$, and which sends a morphism $\tau : A \to A_0$ to $W' \widehat{\otimes}_W \tau$. We define X and Y to be the restrictions of $\widehat{X}$ and $\widehat{Y}$ to the subcategories $\mathcal{C}'$ and $\mathcal{C}$ of Artinian objects in $\widehat{\mathcal{C}}'$ and $\widehat{\mathcal{C}}$, respectively.

LEMMA 8.2. — *The functor* $\widehat{X}$ *is right adjoint to* $\widehat{Y}$, *and* X *is right adjoint to* Y.

Proof. — We prove only the first statement, as the second is similar. For each pair of objects $A \in \mathrm{Ob}(\widehat{\mathcal{C}})$ and $B \in \mathrm{Ob}(\widehat{\mathcal{C}}')$, we define a set map

$$(8.1) \qquad \mu_{A,B} : \mathrm{Hom}_{\widehat{\mathcal{C}}'}\big(\widehat{Y}(A), B\big) \longrightarrow \mathrm{Hom}_{\widehat{\mathcal{C}}}\big(A, \widehat{X}(B)\big)$$

by sending $f \in \mathrm{Hom}_{\widehat{\mathcal{C}}'}(\widehat{Y}(A), B)$ to the morphism $h \in \mathrm{Hom}_{\widehat{\mathcal{C}}}(A, \widehat{X}(B))$ defined by $h(a) = f(1 \widehat{\otimes} a)$ for $a \in A$. This is injective since the image of the map $A \to \widehat{Y}(A)$ defined by $a \to 1 \widehat{\otimes} a$ generates a dense W'-subalgebra of $\widehat{Y}(A)$. It is surjective, since, given $h \in \mathrm{Hom}_{\widehat{\mathcal{C}}}(A, \widehat{X}(B))$, there is a unique $f \in \mathrm{Hom}_{\widehat{\mathcal{C}}'}(\widehat{Y}(A), B)$ such that $f(w \widehat{\otimes} a) = wh(a)$ for $w \in W'$ and $a \in A$. The bijections $\mu_{A,B}$ are natural with respect to morphisms $A \to A_0$ in $\widehat{\mathcal{C}}$ (resp. $B \to B_0$ in $\widehat{\mathcal{C}}'$). So $\widehat{X}$ is right adjoint to $\widehat{Y}$. $\qquad \square$

Recall that $F_\mathcal{D}$ (resp. $F'_\mathcal{D}$) is the restriction of the deformation functor $\widehat{F}_\mathcal{D}$ (resp. $\widehat{F}'_\mathcal{D}$) to the subcategory $\mathcal{C}$ (resp. $\mathcal{C}'$) of Artinian objects.

LEMMA 8.3. — *The deformation functor* $F'_\mathcal{D} : \mathcal{C}' \to \mathrm{Sets}$ *is isomorphic to the composite functor* $F_\mathcal{D} \circ X$.

Proof. — We first consider the case $F_{\mathcal{D}} = F$ and $F'_{\mathcal{D}} = F'$, i.e. $\mathcal{D}$ is empty. We define a natural transformation $\nu : F \circ X \to F'$ in the following way. Suppose B is an object in $\mathcal{C}'$, and that $[(M^\bullet, \phi)]$ is an element of $F(X(B))$, so that $[(M^\bullet, \phi)]$ is the isomorphism class of a quasi-lift $(M^\bullet, \phi)$ of $V^\bullet$ over $X(B)$. Define $\nu([(M^\bullet, \phi)])$ to be the isomorphism class of the quasi-lift $(M'^\bullet, \phi')$ of $V'^\bullet = k' \widehat{\otimes}^L_k V^\bullet$ defined by $M'^\bullet = B \widehat{\otimes}^L_{X(B)} M^\bullet$ and $\phi' = k' \widehat{\otimes}^L_k \phi$. One defines ν on morphisms in the obvious way.

We must now define a natural transformation $\zeta : F' \to F \circ X$ such that $\nu \circ \zeta$ and $\zeta \circ \nu$ are isomorphic to the identity functor. For this we suppose $[(L^\bullet, \psi)]$ is an element of $F'(B)$, where $(L^\bullet, \psi)$ is a quasi-lift of $V'^\bullet$ over B. By Corollary 3.6 (i), there exists a closed normal subgroup Δ of finite index in G so that we can assume the following. The complex $L^\bullet$ (resp. $V^\bullet$) is a bounded above complex of abstractly free finitely generated $[B(G/\Delta)]$-modules (resp. $[k(G/\Delta)]$-modules), and

$$(8.2) \qquad \psi : k' \widehat{\otimes}^L_B L^\bullet = k' \otimes_B L^\bullet \longrightarrow V'^\bullet = k' \widehat{\otimes}^L_k V^\bullet = k' \otimes_k V^\bullet$$

is given by an isomorphism in $D^-([[k'G]])$. By Lemma 3.4, we can assume that ψ is inflated from an isomorphism in $D^-([k'(G/\Delta_1)])$ for a closed normal subgroup Δ_1 of finite index in G with $\Delta_1 \subseteq \Delta$. We now find a bounded above complex $L_1^\bullet$ (resp. $V_1^\bullet$) of abstractly free finitely generated $[B(G/\Delta_1)]$-modules (resp. $[k(G/\Delta_1)]$-modules), together with a quasi-isomorphism $L_1^\bullet \to \mathrm{Inf}_{G/\Delta}^{G/\Delta_1}(L^\bullet)$ in $C^-([B(G/\Delta_1)])$ (resp. $V_1^\bullet \to \mathrm{Inf}_{G/\Delta}^{G/\Delta_1}(V^\bullet)$ in $C^-([k(G/\Delta_1)])$). Using ψ, we obtain an isomorphism

$$\psi_1 : k' \otimes_B L_1^\bullet \longrightarrow k' \otimes_k V_1^\bullet$$

in $D^-([k'(G/\Delta_1)])$. We now replace Δ by Δ_1, $L^\bullet$ by $L_1^\bullet$, $V^\bullet$ by $V_1^\bullet$, and ψ by ψ_1. Then $k' \otimes_B L^\bullet$ and $k' \otimes_k V^\bullet$ are bounded above complexes of abstractly free finitely generated $[k'(G/\Delta)]$-modules. So ψ in (8.2) can be taken to be a quasi-isomorphism in $C^-([k'(G/\Delta)])$. By Remark 5.2, we can add to $L^\bullet$ an acyclic complex of abstractly free finitely generated $[B(G/\Delta)]$-modules so as to be able to assume that ψ is surjective on terms. By Lemma 14.4, there exists a bounded above complex $L_0^\bullet$ of abstractly free finitely generated $[B(G/\Delta)]$-modules and a quasi-isomorphism $\tau : L^\bullet \to L_0^\bullet$ in $C^-([B(G/\Delta)])$ so that there is an isomorphism $\pi : k' \otimes_B L_0^\bullet \to k' \otimes_k V^\bullet$ in $C^-([k'(G/\Delta)])$ with $\pi \circ (k' \otimes_B \tau) = \psi$. We replace $L^\bullet$ by $L_0^\bullet$ and ψ by π so as to be able to assume that ψ in (8.2) is an isomorphism in $C^-([k'(G/\Delta)])$.

Define $M^\bullet$ to be the complex of abelian groups formed by the subcomplex of $L^\bullet$ which maps to $1 \otimes_k V^\bullet$ under the composition of the

natural map $L^\bullet \to k' \otimes_B L^\bullet$ with ψ. Since $V^\bullet$ is a complex of abstractly free finitely generated $[k(G/\Delta)]$-modules, we see that $M^\bullet$ is a complex of abstractly free finitely generated $[X(B)(G/\Delta)]$-modules. Furthermore, the map $b \otimes m \mapsto bm$ gives an isomorphism of complexes $B \otimes_{X(B)} M^\bullet \to L^\bullet$, and ψ induces an isomorphism of complexes

$$\phi : k \otimes_{X(B)} M^\bullet \longrightarrow 1 \otimes_k V^\bullet = V^\bullet.$$

One checks, using Lemma 3.4, Corollary 3.6, and Lemmas 14.1 and 14.4, that the deformation $[(M^\bullet, \phi)] \in F(X(B))$ defined by $(M^\bullet, \phi)$ depends only on the deformation $[(L^\bullet, \psi)] \in F'(B)$ defined by $(L^\bullet, \psi)$. Moreover, the construction of $[(M^\bullet, \phi)]$ is compatible with tensoring with morphisms in $\mathcal{C}'$. By defining $\zeta([(L^\bullet, \psi)]) = [(M^\bullet, \phi)]$, we arrive at a natural transformation $\zeta : F' \to F \circ X$ of functors. The compositions $\zeta \circ \nu$ and $\nu \circ \zeta$ are isomorphic to the identity functor, so F' and $F \circ X$ are isomorphic.

To show F'^{fl} and $F^{\mathrm{fl}} \circ X$ are isomorphic, note that since W' is faithfully flat over W, ν sends proflat deformations to proflat deformations. Concerning ζ, if $[(L^\bullet, \psi)]$ defines a proflat deformation, then $L^\bullet$ splits as a complex of B-modules. Such a splitting gives a splitting of $M^\bullet$ as a complex of $X(B)$-modules, so $[(M^\bullet, \phi)]$ is a proflat deformation. Hence ν and ζ give rise to an isomorphism between F'^{fl} and $F^{\mathrm{fl}} \circ X$. $\qquad\square$

Proof of Theorem 8.1. — We prove the statements in the Theorem when $\mathcal{D}$ is empty, i.e. concerning $R(G, V'^\bullet)$, since the arguments for $R^{\mathrm{fl}}(G, V'^\bullet)$ are similar. Because the functor $\widehat{F}'$ is continuous with versal deformation ring $R(G, V'^\bullet)$, it will suffice to show that $R = W' \widehat{\otimes}_W R(G, V^\bullet)$ is a pro-representable hull for the functor F'. Let $h_R : \mathcal{C}' \to$ Sets be the covariant representation functor associated to R. We have a morphism of functors $\pi : h_R \to F'$ which for $B \in \mathrm{Ob}(\mathcal{C}')$ sends $\gamma \in h_R(B) = \mathrm{Hom}_{\widehat{\mathcal{C}'}}(R, B)$ to the deformation in $F'(B)$ associated to $B \widehat{\otimes}^L_{R, \gamma}(R \widehat{\otimes}^L_{R(G, V^\bullet)} U(G, V^\bullet))$. We must show π is smooth and that

$$(8.3) \qquad \pi_{k'[\varepsilon]} : h_R(k'[\varepsilon]) \longrightarrow F'(k'[\varepsilon])$$

is bijective when $k'[\varepsilon]$ is the ring of dual numbers over k'.

To show smoothness, suppose $B \to C$ is a surjective morphism in $\mathcal{C}'$. We need to show that the map

$$(8.4) \qquad h_R(B) \longrightarrow h_R(C) \times_{F'(C)} F'(B)$$

induced by $\pi : h_R \to F'$ is surjective. By Lemmas 8.2 and 8.3, we can identify (8.4) with the map

$$(8.5) \qquad h_{R(G,V^\bullet)}\big(X(B)\big) \longrightarrow h_{R(G,V^\bullet)}\big(X(C)\big) \times_{F(X(C))} F\big(X(B)\big)$$

associated to the covariant representation functor $h_{R(G,V^\bullet)} : \mathcal{C} \to$ Sets and the surjection $X(B) \to X(C)$ in $\mathcal{C}$. Since $R(G,V^\bullet)$ is the versal deformation ring of $V^\bullet$, (8.5) is surjective, so (8.4) is surjective.

We can by the same argument identify (8.3) with the map

$$(8.6) \qquad h_{R(G,V^\bullet)}\big(X(k'[\varepsilon])\big) \longrightarrow F\big(X(k'[\varepsilon])\big)$$

which is surjective, since (8.5) is surjective when $B = k'[\varepsilon]$ and $C = k'$. Since k' is a finite extension of k, $X(k'[\varepsilon]) = k + k'\varepsilon$ is the inverse limit of a set of residue homomorphisms $r_j = r : k[\varepsilon] \to k$, $j = 1, \ldots, t$, where $t = \dim_k \mathrm{Hom}_k(k', k)$,

$$
\begin{array}{c}
k[\varepsilon] \xrightarrow{\ r_1\ } \\
k[\varepsilon] \xrightarrow{\ r_2\ } \quad k \\
\vdots \\
k[\varepsilon] \xrightarrow{\ r_t\ }
\end{array}
$$

indexed by a k-basis for $\mathrm{Hom}_k(k', k)$. Because F satisfies Schlessinger's criterion (H2) (see Lemmas 5.3 and 5.4), we can apply (H2) repeatedly to obtain that F commutes with the finite inverse limit representing $X(k'[\varepsilon]) = k + k'\varepsilon$. This reduces the proof of showing that (8.6) is injective to showing that

$$h_{R(G,V^\bullet)}\big(k[\varepsilon]\big) \longrightarrow F\big(k[\varepsilon]\big)$$

is injective. Because $R(G,V^\bullet)$ is the versal deformation ring of $V^\bullet$, this completes the proof of Theorem 8.1.

9. One-term and two-term complexes.

In this section, we consider the case when $V^\bullet$ has one or two non-zero cohomology groups.

PROPOSITION 9.1. — *Suppose G has finite pseudocompact cohomology, and that $V^\bullet$ has exactly one non-zero cohomology group C, which has finite k-dimension. Then $R(G,V^\bullet)$ coincides with the versal deformation ring $R(G,C)$ considered by Mazur [19]; in particular, $R(G,V^\bullet) = R^{\mathrm{fl}}(G,V^\bullet)$. The groups $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet)$ and $\mathrm{Hom}_{[[kG]]}(C,C) = \mathrm{Hom}_{[kG]}(C,C)$ are isomorphic.*

Proof. — It will suffice to show that the functor $\widehat{F} : \widehat{\mathcal{C}} \to$ Sets defined by the isomorphism classes of quasi-lifts of $V^\bullet$ coincides with Mazur's functor $\widehat{F}_C$ defined by isomorphism classes of lifts of C [19]. Because $\widehat{F}$ and $\widehat{F}_C$ are continuous (see Proposition 7.2 and [20, §20 Prop. 1]), it will suffice to show that the restrictions F and F_C of these functors to $\mathcal{C}$ are isomorphic. Without loss of generality we can assume $C = \mathrm{H}^0(V^\bullet)$.

If R is an Artinian object in $\mathrm{Ob}(\mathcal{C})$, then every quasi-lift $M^\bullet$ of $V^\bullet$ over R has non-zero cohomology only in dimension 0 by Lemma 3.1. Hence $M^\bullet$ is, as a quasi-lift, isomorphic to $M'^\bullet$ where $M'^0 = \mathrm{H}^0(M^\bullet)$ and $M'^i = 0$ otherwise. Since, by Lemma 3.8, $M^\bullet$ has finite pseudocompact R-tor dimension at 0, it follows by Remark 2.6 that M'^0 has projective dimension 0 as abstract R-module. Hence M'^0 is an abstractly free R-module because R is local Artinian. Therefore $k \otimes_R M'^0$ is isomorphic to $C = \mathrm{H}^0(V^\bullet)$ since $k \otimes_R^{\boldsymbol{L}} M^\bullet$ is isomorphic to $V^\bullet$ in the derived category. We have now shown that M'^0 is a lift of C over R, and the isomorphism class of M'^0 as a lift of C over R determines the isomorphism class of $M^\bullet$ in $D^-([[RG]])$. Conversely, suppose L is a lift of C over R. By Mazur's definition of lifts [19], this means that L is an abstractly free R-module of rank equal to $\dim_k C$. Since R is Artinian, this implies that L is a discrete $[[RG]]$-module of finite length, hence a pseudocompact $[[RG]]$-module. Thus the complex $L^\bullet$ with $L^0 = L$ and $L^i = 0$ for $i \neq 0$ is a quasi-lift of $V^\bullet$ over R. This shows F and F_C are isomorphic functors. $\qquad \square$

We now consider $V^\bullet$ having two non-zero cohomology groups. Without loss of generality, we can suppose these groups are $U_0 = \mathrm{H}^0(V^\bullet)$ and $U_{-n} = \mathrm{H}^{-n}(V^\bullet)$ for some $n > 0$. If M is a pseudocompact $[[kG]]$-module, we will also regard M as a complex concentrated in dimension 0. By [17, Cor. I.6.5], $\mathrm{Hom}_{D^-([[kG]])}(M, T^j(M')) = \mathrm{Ext}^j_{[[kG]]}(M, M')$ for all integers j, and this group is 0 if $j < 0$. We first want to find necessary and sufficient conditions for $V^\bullet$ to satisfy $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet) = k$. We need the following definition.

DEFINITION 9.2. — Suppose $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$, and let $\beta : M^\bullet \to N^\bullet$ be a morphism in $D^-([[RG]])$. Then β can be represented by a pair of morphisms in $C^-([[RG]])$ of the form

$$
\begin{array}{ccc}
 & Z^\bullet & \\
{\scriptstyle \beta_1}\swarrow & & \searrow {\scriptstyle \beta_2} \\
M^\bullet & & N^\bullet
\end{array}
$$

where β_1 is a quasi-isomorphism. We get a triangle in $K^-([[RG]])$

$$
\begin{array}{ccccccc}
Z^\bullet & \xrightarrow{\beta_2} & N^\bullet & \longrightarrow & C(\beta_2)^\bullet & \longrightarrow & T(Z^\bullet) \\
\downarrow{\scriptstyle\beta_1} & & \| & & \| & & \downarrow{\scriptstyle T(\beta_1)} \\
M^\bullet & & N^\bullet & & C(\beta_2)^\bullet & & T(M^\bullet)
\end{array}
$$

where the downward arrows are quasi-isomorphisms in $C^-([[RG]])$. Thus we get a triangle

$$
M^\bullet \xrightarrow{\beta} N^\bullet \longrightarrow C(\beta_2)^\bullet \longrightarrow T(M^\bullet)
$$

in $D^-([[RG]])$. Since the isomorphism class of $C(\beta_2)^\bullet$ in $D^-([[RG]])$ only depends on β by [17, Prop. I.1.1], but not on the choice of β_2, we call $C(\beta_2)^\bullet$ the *mapping cone of β* and denote it by $C(\beta)^\bullet$.

PROPOSITION 9.3. — *Suppose $V^\bullet$ has exactly two non-zero cohomology groups $U_0 = \mathrm{H}^0(V^\bullet)$ and $U_{-n} = \mathrm{H}^{-n}(V^\bullet)$, for some $n > 0$, both of finite k-dimension. Then $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet) = k$ if and only if all of the following conditions hold*:

(i) $\mathrm{Ext}^n_{D^-([[kG]])}(U_0, U_{-n}) = 0$.

(ii) *$V^\bullet$ defines a nontrivial element*

$$
\beta \in \mathrm{Hom}_{D^-([[kG]])}\big(U_0, T^{n+1}(U_{-n})\big) = \mathrm{Ext}^{n+1}_{D^-([[kG]])}(U_0, U_{-n})
$$

so that $V^\bullet \cong T^{-1}(C(\beta)^\bullet)$ in $D^-([[kG]])$.

(iii) *The composition with β induces injective maps*

$$
\mathrm{Hom}_{D^-([[kG]])}(U_0, U_0) \xrightarrow{\beta\circ} \mathrm{Hom}_{D^-([[kG]])}\big(U_0, T^{n+1}(U_{-n})\big),
$$

$$
\mathrm{Hom}_{D^-([[kG]])}\big(T^{n+1}(U_{-n}), T^{n+1}(U_{-n})\big)
$$
$$
\xrightarrow{\circ\beta} \mathrm{Hom}_{D^-([[kG]])}\big(U_0, T^{n+1}(U_{-n})\big).
$$

(iv) *In $\mathrm{Ext}^{n+1}_{D^-([[kG]])}(U_0, U_{-n})$,*

$$
k\beta = \big[\beta \circ \mathrm{Hom}_{D^-([[kG]])}(U_0, U_0)\big]
$$
$$
\cap \big[\mathrm{Hom}_{D^-([[kG]])}(T^{n+1}(U_{-n}), T^{n+1}(U_{-n})) \circ \beta\big].
$$

Proof. — Without loss of generality, we can assume that

$$
V^\bullet = \cdots \to 0 \to V^{-n} \longrightarrow V^{-n+1} \longrightarrow \cdots \longrightarrow V^{-1} \longrightarrow V^0 \to 0 \to \cdots.
$$

As in (6.14), we have a short exact sequence of complexes

$$
0 \to T^n(U_{-n}) \xrightarrow{\iota} V^\bullet \xrightarrow{\widetilde{\pi}} \widetilde{V}^\bullet \to 0
$$

which, by [17, proof of Prop. I.6.1], defines a triangle

$$
T^n(U_{-n}) \xrightarrow{\iota} V^\bullet \xrightarrow{\widetilde{\pi}} \widetilde{V}^\bullet \xrightarrow{\widetilde{\beta}} T^{n+1}(U_{-n}).
$$

Since there is a quasi-isomorphism $q : \widetilde{V}^\bullet \to U_0$, we obtain a triangle

$$
(9.1) \qquad T^n(U_{-n}) \xrightarrow{\iota} V^\bullet \xrightarrow{\pi} U_0 \xrightarrow{\beta} T^{n+1}(U_{-n}),
$$

and $V^\bullet$ is isomorphic to $T^{-1}(C(\beta)^\bullet)$ in $D^-([[kG]])$. We obtain the following diagram with exact rows and columns, using long exact Hom sequences associated to the triangle (9.1):

$$
(9.2)
$$

where $\mathrm{Hom} = \mathrm{Hom}_{D^-([[kG]])}$. Since the terms of $V^\bullet$ are concentrated between dimensions $-n$ and 0, it follows that $\mathrm{Hom}(T^{n+1}(U_{-n}), V^\bullet) = 0$. Since $-n < 0$, we have

$$
\mathrm{Hom}(T^n(U_{-n}), U_0) = \mathrm{Ext}^{-n}_{[[kG]]}(U_{-n}, U_0) = 0.
$$

Hence we have isomorphisms in $D^-([[kG]])$

$$
(9.3) \qquad
\begin{cases}
\mathrm{Hom}(U_0, U_0) \xrightarrow{\circ\pi} \mathrm{Hom}(V^\bullet, U_0), \\[2mm]
\mathrm{Hom}\big(T^n(U_{-n}), T^n(U_{-n})\big) \xrightarrow{\iota\circ} \mathrm{Hom}\big(T^n(U_{-n}), V^\bullet\big).
\end{cases}
$$

Suppose first that $\mathrm{Hom}(V^\bullet, V^\bullet) = k$. It follows from the triangle (9.1) that β is nontrivial, which is condition (ii) of Proposition 9.3. The triangle (9.1) also shows that if π (resp. ι) are zero in $D^-([[kG]])$, then $T^n(U_{-n}) \cong T^{-1}(U_0) \oplus V^\bullet$ (resp. $U_0 \cong V^\bullet \oplus T^{n+1}(U_{-n})$) in $D^-([[kG]])$. Since this is impossible, it follows that the maps $\mathrm{Hom}(V^\bullet, V^\bullet) \xrightarrow{\pi\circ} \mathrm{Hom}(V^\bullet, U_0)$ and $\mathrm{Hom}(V^\bullet, V^\bullet) \xrightarrow{\circ\iota} \mathrm{Hom}(T^n(U_{-n}), V^\bullet)$ are injective. Hence diagram (9.2) implies

$$\mathrm{Hom}(V^\bullet, T^n(U_{-n})) = 0 = \mathrm{Hom}(U_0, V^\bullet).$$

This is the case if and only if conditions (i) and (iii) of Proposition 9.3 hold. The kernel of the homomorphism

$$\mathrm{Hom}(V^\bullet, U_0) \xrightarrow{\beta\circ} \mathrm{Hom}(V^\bullet, T^{n+1}(U_{-n}))$$

is equal to $k\pi$, and is by the first isomorphism in (9.3) isomorphic to the kernel of the composite homomorphism

$$\mathrm{Hom}(U_0, U_0) \xrightarrow{\gamma} \mathrm{Hom}(V^\bullet, T^{n+1}(U_{-n}))$$

where γ is the composition of $\beta\circ$ and $\circ\pi$. In particular, $\mathrm{Ker}(\gamma)$ contains the scalar multiplications in $\mathrm{Hom}(U_0, U_0)$. Because $\mathrm{Hom}(U_0, U_0) \xrightarrow{\beta\circ} \mathrm{Hom}(U_0, T^{n+1}(U_{-n}))$ is injective, the kernel of γ is isomorphic to the k-subspace

$$\big[\beta \circ \mathrm{Hom}(U_0, U_0)\big] \cap \big[\mathrm{Hom}(T^{n+1}(U_{-n}), T^{n+1}(U_{-n})) \circ \beta\big]$$

inside $\mathrm{Hom}(U_0, T^{n+1}(U_{-n}))$. This k-vector space contains the one-dimensional k-space $k\beta$, which implies condition (iv) of Proposition 9.3.

Suppose now that conditions (i), (ii), (iii) and (iv) hold. We need to show that $\mathrm{Hom}(V^\bullet, V^\bullet) = k$. The injectivity of the first morphism in condition (iii) implies that the image of $\mathrm{Hom}(U_0, V^\bullet)$ under $\pi\circ$ is zero. Hence diagram (9.2) implies that the kernel of $\pi\circ$ is $\mathrm{Hom}(U_0, V^\bullet)$, and thus

$$\mathrm{Hom}(U_0, T^n(U_{-n})) \xrightarrow{\iota\circ} \mathrm{Hom}(U_0, V^\bullet)$$

is an isomorphism. Since $\mathrm{Hom}(U_0, T^n(U_{-n})) = \mathrm{Ext}^n(U_0, U_{-n}) = 0$ by condition (i), it follows that $\mathrm{Hom}(U_0, V^\bullet) = 0$. Similarly, using the injectivity of the second morphism in condition (iii), one obtains $\mathrm{Hom}(V^\bullet, T^n(U_{-n})) = 0$. Hence the two morphisms

$$\mathrm{Hom}(V^\bullet, V^\bullet) \xrightarrow{\pi\circ} \mathrm{Hom}(V^\bullet, U_0), \ \mathrm{Hom}(U_0, U_0) \xrightarrow{\beta\circ} \mathrm{Hom}(U_0, T^{n+1}(U_{-n}))$$

are injective. Let

$$U = \big[\beta \circ \mathrm{Hom}(U_0, U_0)\big] \cap \big[\mathrm{Hom}(T^{n+1}(U_{-n}), T^{n+1}(U_{-n})) \circ \beta\big],$$

and let ψ be the inverse of the first isomorphism in (9.3). Then we have a diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(V^\bullet, V^\bullet) & \xrightarrow{\pi\circ} & \mathrm{Hom}(V^\bullet, U_0) & \xrightarrow{\beta\circ} & \beta \circ \mathrm{Hom}(V^\bullet, U_0) & \rightarrow & 0 \\
 & & & & \circ\pi \uparrow \downarrow \psi & & \uparrow \circ\pi & & \\
0 & \longrightarrow & & & \mathrm{Hom}(U_0, U_0) & \xrightarrow{\beta\circ} & \beta \circ \mathrm{Hom}(V^\bullet, U_0) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & U & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & & \\
\end{array}
$$

and $\mathrm{Hom}(V^\bullet, V^\bullet)$ is isomorphic to U, which is $k\beta$ by condition (iv). Hence $\mathrm{Hom}(V^\bullet, V^\bullet) = k$. □

COROLLARY 9.4. — *In the situation of Proposition 9.3, if the endomorphism rings of U_0 and U_{-n} are both given by scalars, then $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet) = k$ if and only if*

(i) $\mathrm{Ext}^n_{[[kG]]}(U_0, U_{-n}) = 0$, *and*

(ii) $V^\bullet \cong T^{-1}(C(\beta)^\bullet)$ *in* $D^-([[kG]])$ *for a nontrivial element β in* $\mathrm{Ext}^{n+1}_{[[kG]]}(U_0, U_{-n})$, *where $C(\beta)^\bullet$ is as in Definition 9.2.*

Example 9.5. — Suppose $p > 2$, k is an algebraically closed field of characteristic p, and let $G = (\mathbb{Z}/p^2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ be a dihedral group of order $2p^2$. Then $[kG]$ is its principal block, which is a block with cyclic defect groups isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$. Hence the corresponding Brauer tree B has two edges and multiplicity $m = \frac{1}{2}(p^2 - 1)$ (for background on Brauer trees we refer to [1]):

$$\circ \xrightarrow{\ S_1\ } \bullet \xrightarrow{\ S_2\ } \circ$$
$$ m$$

where S_1 and S_2 are the two one-dimensional non-isomorphic $[kG]$-modules. The multiplicity $m = \frac{1}{2}(p^2 - 1)$ is greater than 3. We define $U_0 = S_2$ to be simple, and U_{-1} to be uniserial with radical series (S_2, S_1, S_2, S_1). Then the stable endomorphism rings are $\underline{\mathrm{End}}_{[kG]}(U_0) = k$ and $\underline{\mathrm{End}}_{[kG]}(U_{-1}) = k^2$.

We claim

$$(9.4) \qquad \mathrm{Ext}^1_{[kG]}(U_0, U_{-1}) = \underline{\mathrm{Hom}}_{[kG]}\big(\Omega(U_0), U_{-1}\big) = 0$$

where Ω denotes the Heller operator (see e.g. [1]). This follows, since $\Omega(U_0)$ is uniserial with descending radical series $(S_1, S_2)^{\frac{1}{2}(p^2-1)}$. So $\mathrm{Hom}_{[kG]}(\Omega(U_0), U_{-1}) = k^2$ is generated by the compositions

$$f_1 : \Omega(U_0) \xrightarrow{\pi_1} S_1 \xrightarrow{\iota_1} U_{-1} \quad \text{and} \quad f_2 : \Omega(U_0) \xrightarrow{\pi_2} \begin{pmatrix} S_1 \\ S_2 \\ S_1 \end{pmatrix} \xrightarrow{\iota_2} U_{-1}$$

where π_i are natural surjections and ι_i are natural injections. Both f_1 and f_2 factor through the projective cover P_2 of S_2, which proves (9.4). We have

$$\mathrm{Ext}^2_{[kG]}(U_0, U_{-1}) = \underline{\mathrm{Hom}}_{[kG]}\big(\Omega^2(U_0), U_{-1}\big) = \mathrm{Hom}_{[kG]}(S_1, U_{-1}) = k.$$

Hence conditions (i), (ii), (iii) and (iv) of Proposition 9.3 are satisfied for

$$V^\bullet : \quad \cdots \to 0 \to \begin{pmatrix} S_1 \\ S_2 \\ S_1 \\ S_2 \\ S_1 \end{pmatrix} \xrightarrow{\delta} \begin{pmatrix} S_2 \\ S_1 \end{pmatrix} \to 0 \to \cdots$$

where δ is a non-zero homomorphism which factors through S_1.

Note that since the stable endomorphism ring of U_{-1} is not k, it is not clear whether U_{-1} has a universal deformation ring. On the other hand, by Theorem 2.14, $V^\bullet$ has a universal deformation ring.

We now give a description of the tangent space of the proflat deformation functor $\widehat{F}^{\mathrm{fl}}$ using Remark 2.17 (i).

PROPOSITION 9.6. — Suppose $V^\bullet$ has exactly two non-zero cohomology groups $U_0 = \mathrm{H}^0(V^\bullet)$ and $U_{-n} = \mathrm{H}^{-n}(V^\bullet)$ for some $n > 0$ so that we have a triangle

$$(9.5) \qquad T^n(U_{-n}) \xrightarrow{\iota} V^\bullet \xrightarrow{\pi} U_0 \xrightarrow{\beta} T^{n+1}(U_{-n}).$$

    (i) If $n \geq 2$, then $t_{F^{\mathrm{fl}}} = t_F$.

    (ii) If $n = 1$, then $t_{F^{\mathrm{fl}}}$ is the subspace of $t_F = \mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet, V^\bullet)$ $= \mathrm{Hom}_{D^-([[kG]])}(V^\bullet, T(V^\bullet))$ consisting of those elements $f \in t_F$ with $T(\pi) \circ f \circ \iota = 0$ in $\mathrm{Hom}_{D^-([[kG]])}(T(U_{-1}), T(U_0))$.

    (iii) Suppose $n = 1$ and $\mathrm{Ext}^3_{[[kG]]}(U_0, U_{-1}) = 0 = \mathrm{Ext}^4_{[[kG]]}(U_0, U_{-1})$. Then $t_{F^{\mathrm{fl}}} = t_F$ if and only if β from (9.5) defines a map

$$\mathrm{Ext}^1_{D^-([[kG]])}\big(T(U_{-1}), U_0\big) \xrightarrow{\circ T^{-1}(\beta)} \mathrm{Ext}^1_{D^-([[kG]])}\big(T^{-1}(U_0), U_0\big)$$

which is injective when restricted to the kernel of the map

$$\mathrm{Ext}^1_{D^-([[kG]])}\big(T(U_{-1}), U_0\big) \xrightarrow{T(\beta)\circ} \mathrm{Ext}^1_{D^-([[kG]])}\big(T(U_{-1}), T^2(U_{-1})\big).$$

This is the case if and only if

$$\mathrm{Ext}^1_{D^-([[kG]])}\big(T(U_{-1}), U_0\big) \xrightarrow{T(\beta)\circ} \mathrm{Ext}^1_{D^-([[kG]])}\big(T(U_{-1}), T^2(U_{-1})\big)$$

is injective when restricted to the kernel of the map

$$\mathrm{Ext}^1_{D^-([[kG]])}\big(T(U_{-1}), U_0\big) \xrightarrow{\circ T^{-1}(\beta)} \mathrm{Ext}^1_{D^-([[kG]])}\big(T^{-1}(U_0), U_0\big).$$

Proof. — According to Remark 2.17 (i), the tangent space $t_{F^{\mathrm{fl}}}$ consists of those elements $f \in t_F$ which induce the trivial map on cohomology. In other words, the k-vector space maps $f^i : \mathrm{H}^i(V^\bullet) \to \mathrm{H}^{i+1}(V^\bullet)$ which are induced by f have to be zero for all i. When $n \geq 2$, then $\mathrm{H}^i(V^\bullet) = 0$ or $\mathrm{H}^{i+1}(V^\bullet) = 0$ for all i. Thus all f^i are zero, which implies part (i). If $n = 1$, we also get $f^i = 0$ unless $i = -1$. For parts (ii) and (iii), we consider the following diagram with exact rows and columns obtained from long exact Hom sequences associated to the triangle (9.5):

$$(9.6)$$

$$
\begin{array}{ccccc}
 & & & & \mathrm{Ext}^1(U_0, T^2(U_{-1})) \\
 & & & & \downarrow \\
\mathrm{Ext}^1(V^\bullet, V^\bullet) & \xrightarrow{T(\pi)\circ} & \mathrm{Ext}^1(V^\bullet, U_0) & \xrightarrow{T(\beta)\circ} & \mathrm{Ext}^1(V^\bullet, T^2(U_{-1})) \\
\downarrow{\scriptstyle\circ\iota} & & \downarrow{\scriptstyle\circ\iota} & & \downarrow{\scriptstyle\circ\iota} \\
\mathrm{Ext}^1(T(U_{-1}), V^\bullet) & \xrightarrow{T(\pi)\circ} & \mathrm{Ext}^1(T(U_{-1}), U_0) & \xrightarrow{T(\beta)\circ} & \mathrm{Ext}^1(T(U_{-1}), T^2(U_{-1})) \\
\downarrow{\scriptstyle\circ T^{-1}(\beta)} & & \downarrow{\scriptstyle\circ T^{-1}(\beta)} & & \downarrow \\
\mathrm{Ext}^1(T^{-1}(U_0), T(U_{-1})) \longrightarrow \mathrm{Ext}^1(T^{-1}(U_0), V^\bullet) & \xrightarrow{T(\pi)\circ} & \mathrm{Ext}^1(T^{-1}(U_0), U_0) & \longrightarrow & \mathrm{Ext}^1(T^{-1}(U_0), T^2(U_{-1}))
\end{array}
$$

As before, $\mathrm{Hom} = \mathrm{Hom}_{D^-([[kG]])}$ and $\mathrm{Ext}^i = \mathrm{Ext}^i_{D^-([[kG]])}$. Then $\mathrm{Ext}^1(T(U_{-1}), U_0) = \mathrm{Hom}(U_{-1}, U_0)$, and $f \in t_F$ induces

$$T(\pi) \circ f \circ \iota : \mathrm{H}^{-1}(V^\bullet) \longrightarrow \mathrm{H}^0(V^\bullet)$$

which gives the vector space map f^{-1} in dimension -1. This implies part (ii). For part (iii), we notice that

$$\mathrm{Ext}^1\big(U_0, T^2(U_{-1})\big) = \mathrm{Ext}^1\big(T^{-1}(U_0), T(U_{-1})\big) = \mathrm{Ext}^3(U_{-1}, U_0) = 0,$$

and $\mathrm{Ext}^1(T^{-1}(U_0), T^2(U_{-1})) = \mathrm{Ext}^4(U_0, U_{-1}) = 0$. This means that

$$(9.7) \quad \begin{cases} \mathrm{Ext}^1\big(T^{-1}(U_0), V^\bullet\big) \xrightarrow{T(\pi)\circ} \mathrm{Ext}^1\big(T^{-1}(U_0), U_0\big), \\ \mathrm{Ext}^1\big(V^\bullet, T^2(U_{-1})\big) \xrightarrow{\circ\iota} \mathrm{Ext}^1\big(T(U_{-1}), T^2(U_{-1})\big) \end{cases}$$

are isomorphisms. Hence, $T(\pi) \circ f \circ \iota = 0$ for all $f \in \mathrm{Ext}^1(V^\bullet, V^\bullet)$, if and only if $T(\pi) \circ g = 0$ for all $g \in \mathrm{Ext}^1(T(U_{-1}), V^\bullet)$ with $g \circ T^{-1}(\beta) = 0$. Because of (9.7), $g \circ T^{-1}(\beta) = 0$ if and only if $T(\pi) \circ g \circ T^{-1}(\beta) = 0$. We conclude that $T(\pi) \circ f \circ \iota = 0$ for all $f \in \mathrm{Ext}^1(V^\bullet, V^\bullet)$, if and only if

$$\mathrm{Ext}^1\big(T(U_{-1}), U_0\big) \xrightarrow{\circ T^{-1}(\beta)} \mathrm{Ext}^1\big(T^{-1}(U_0), U_0\big)$$

is injective when restricted to $\{T(\pi) \circ g \mid g \in \mathrm{Ext}^1(T(U_{-1}), V^\bullet)\}$, which is the kernel of the map

$$\mathrm{Ext}^1\big(T(U_{-1}), U_0\big) \xrightarrow{T(\beta)\circ} \mathrm{Ext}^1\big(T(U_{-1}), T^2(U_{-1})\big).$$

Similarly, $T(\pi) \circ f \circ \iota = 0$ for all $f \in \mathrm{Ext}^1(V^\bullet, V^\bullet)$, if and only if $h \circ \iota = 0$ for all $h \in \mathrm{Ext}^1(V^\bullet, U_0)$ with $T(\beta) \circ h \circ \iota = 0$. This is the case, if and only if

$$\mathrm{Ext}^1\big(T(U_{-1}), U_0\big) \xrightarrow{T(\beta)\circ} \mathrm{Ext}^1\big(T(U_{-1}), T^2(U_{-1})\big)$$

is injective when restricted to $\{h \circ \iota \mid h \in \mathrm{Ext}^1(V^\bullet, U_0)\}$, which is the kernel of the map

$$\mathrm{Ext}^1\big(T(U_{-1}), U_0\big) \xrightarrow{\circ T^{-1}(\beta)} \mathrm{Ext}^1\big(T^{-1}(U_0), U_0\big). \qquad \square$$

PROPOSITION 9.7. — *With the hypotheses of Proposition 9.6, suppose in addition that* U_{-n}, U_0 *have universal deformation rings* R_{-n}, R_0 *and universal deformations* X_{-n}, X_0, *in the sense of Mazur [19], and that*

$$(9.8) \quad \dim_k \mathrm{Ext}^1_{[[kG]]}(U_{-n}, U_{-n}) + \dim_k \mathrm{Ext}^1_{[[kG]]}(U_0, U_0) = \dim_k t_{F^{\mathrm{fl}}}.$$

Suppose furthermore that there exists a proflat quasi-lift $(M^\bullet, \phi)$ *of* $V^\bullet$ *over* $R_{-n} \widehat\otimes_W R_0$ *such that*

$$(9.9) \quad \begin{cases} \mathrm{H}^{-n}(M^\bullet) \cong (R_{-n}\widehat\otimes_W R_0)\widehat\otimes_{R_{-n}} X_{-n}, \\ \mathrm{H}^0(M^\bullet) \cong (R_{-n}\widehat\otimes_W R_0)\widehat\otimes_{R_0} X_0. \end{cases}$$

Then the versal proflat deformation ring $R^{\mathrm{fl}}(G, V^\bullet)$ *is universal and isomorphic to* $R_{-n}\widehat\otimes_W R_0$.

Proof. — Since $(M^\bullet, \phi)$ is a proflat quasi-lift of $V^\bullet$ over $R_{-n}\widehat\otimes_W R_0$, there exists a morphism

$$\alpha : R^{\mathrm{fl}} = R^{\mathrm{fl}}(G, V^\bullet) \longrightarrow R_{-n}\widehat\otimes_W R_0$$

in $\widehat{\mathcal{C}}$ so that $M^\bullet \cong (R_{-n}\widehat\otimes_W R_0)\widehat\otimes^{\boldsymbol{L}}_{R^{\mathrm{fl}}, \alpha} U^{\mathrm{fl}}(G, V^\bullet)$ in $D^-([[(R_{-n}\widehat\otimes_W R_0)G]])$. If $\rho : R_{-n}\widehat\otimes_W R_0 \to k[\varepsilon]$ is a morphism in $\widehat{\mathcal{C}}$, then ρ is equal to the tensor product $\rho_{-n}\widehat\otimes\rho_0$ where $\rho_{-n} : R_{-n} \to k[\varepsilon]$ (resp. $\rho_0 : R_0 \to k[\varepsilon]$) is associated to the lift of $U_{-n} = \mathrm{H}^{-n}(V^\bullet)$ (resp. $U_0 = \mathrm{H}^0(V^\bullet)$) over $k[\varepsilon]$ resulting from the cohomology group $\mathrm{H}^{-n}(k[\varepsilon]\widehat\otimes^{\boldsymbol{L}}_{R_{-n}\widehat\otimes_W R_0, \rho} M^\bullet)$ (resp. $\mathrm{H}^0(k[\varepsilon]\widehat\otimes^{\boldsymbol{L}}_{R_{-n}\widehat\otimes_W R_0, \rho} M^\bullet)$). Because of (9.9) it follows that when ρ ranges over all morphisms $R_{-n}\widehat\otimes_W R_0 \to k[\varepsilon]$ in $\widehat{\mathcal{C}}$, then the pair

$$\left(\mathrm{H}^{-n}\big(k[\varepsilon]\widehat\otimes^{\boldsymbol{L}}_{R_{-n}\widehat\otimes_W R_0, \rho} M^\bullet\big), \mathrm{H}^0\big(k[\varepsilon]\widehat\otimes^{\boldsymbol{L}}_{R_{-n}\widehat\otimes_W R_0, \rho} M^\bullet\big)\right)$$

ranges over all possible pairs of deformations of (U_{-n}, U_0), in the sense of Mazur, over $k[\varepsilon]$. This implies that for each different ρ, $k[\varepsilon]\widehat\otimes^{\boldsymbol{L}}_{R_{-n}\widehat\otimes_W R_0, \rho} M^\bullet$ defines a different proflat deformation of $V^\bullet$ over $k[\varepsilon]$. Thus (9.8) implies that α is surjective.

On the other hand, since R_{-n} and R_0 are universal, there exist unique morphisms $\phi_{-n} : R_{-n} \to R^{\mathrm{fl}}$ and $\phi_0 : R_0 \to R^{\mathrm{fl}}$ in $\widehat{\mathcal{C}}$ such that

$$\mathrm{H}^{-n}(U^{\bullet\mathrm{fl}}) \cong R^{\mathrm{fl}}\widehat\otimes_{R_{-n}, \phi_{-n}} X_{-n} \quad \text{and} \quad \mathrm{H}^0(U^{\bullet\mathrm{fl}}) \cong R^{\mathrm{fl}}\widehat\otimes_{R_0, \phi_0} X_0$$

where $U^{\bullet\mathrm{fl}} = U^{\mathrm{fl}}(G, V^\bullet)$. By Lemma 4.2, we can assume that the terms of $U^{\bullet\mathrm{fl}}$ are topologically free pseudocompact R^{fl}-modules. Because the completed tensor product over W is the coproduct in the category $\widehat{\mathcal{C}}$, we obtain a unique W-algebra homomorphism

$$\beta = \phi_{-n}\widehat\otimes\phi_0 : R_{-n}\widehat\otimes_W R_0 \longrightarrow R^{\mathrm{fl}}.$$

By what we proved above concerning α, we know that as τ ranges over all morphisms $R^{\mathrm{fl}} \to k[\varepsilon]$ in $\widehat{\mathcal{C}}$, the groups

$$\mathrm{H}^j\big(k[\varepsilon]\widehat\otimes^{\boldsymbol{L}}_{R^{\mathrm{fl}}, \tau} U^{\bullet\mathrm{fl}}\big) = k[\varepsilon]\widehat\otimes_{R^{\mathrm{fl}}, \tau}\mathrm{H}^j(U^{\bullet\mathrm{fl}})$$
$$= k[\varepsilon]\widehat\otimes_{R^{\mathrm{fl}}, \tau}(R^{\mathrm{fl}}\widehat\otimes_{R_j, \phi_j} X_j) = k[\varepsilon]\widehat\otimes_{R_j, \tau\circ\phi_j} X_j$$

for $j = -n$ and $j = 0$, range over all possible pairs of deformations of U_{-n} and U_0 over $k[\varepsilon]$. It follows that $\tau \circ (\phi_{-n}\otimes\phi_0) = \tau \circ \beta$ ranges over all possible homomorphisms $R_{-n} \otimes R_0 \to k[\varepsilon]$ in $\widehat{\mathcal{C}}$. However, (9.8) implies that the number of such homomorphisms is equal to the number of $\tau : R^{\mathrm{fl}} \to k[\varepsilon]$ in $\widehat{\mathcal{C}}$. Hence the map $\tau \to \tau \circ \beta$ is bijective, and β must be surjective.

We obtain a surjective continuous W-algebra homomorphism

$$f = \beta \circ \alpha : R^{\mathrm{fl}} \longrightarrow R^{\mathrm{fl}}.$$

This means that f maps the maximal ideal $m_{R^{\mathrm{fl}}}$ onto $m_{R^{\mathrm{fl}}}$, and therefore it induces surjective W-algebra homomorphisms $f_i : R^{\mathrm{fl}}/m^i_{R^{\mathrm{fl}}} \to R^{\mathrm{fl}}/m^i_{R^{\mathrm{fl}}}$ for all i. Since $R^{\mathrm{fl}}/m^i_{R^{\mathrm{fl}}}$ is Artinian, each f_i is in fact an isomorphism. Because $f = \varprojlim_i f_i$, it follows that f is injective. Hence α and β are isomorphisms, $R^{\mathrm{fl}} \cong R_{-n} \widehat{\otimes}_W R_0$, and $M^\bullet \cong U^{\bullet \mathrm{fl}}$.

It remains to show that R^{fl} is universal. Suppose $(N^\bullet, \psi)$ is a proflat quasi-lift of $V^\bullet$ over some object R in $\widehat{\mathcal{C}}$. Then there exists a morphism $\gamma : R^{\mathrm{fl}} \to R$ so that $N^\bullet \cong R \widehat{\otimes}^{\boldsymbol{L}}_{R^{\mathrm{fl}}, \gamma} U^{\bullet \mathrm{fl}}$ in $D^-([[RG]])$. Since R^{fl} is isomorphic to $R_{-n} \widehat{\otimes}_W R_0$, γ is given by a tensor product $\gamma_{-n} \widehat{\otimes} \gamma_0$ where $\gamma_{-n} : R_{-n} \to R$ and $\gamma_0 : R_0 \to R$ in $\widehat{\mathcal{C}}$. We obtain $[[RG]]$-module isomorphisms

$$\mathrm{H}^j(N^\bullet) \cong \mathrm{H}^j(R \widehat{\otimes}^{\boldsymbol{L}}_{R^{\mathrm{fl}}, \gamma} U^{\bullet \mathrm{fl}}) \cong R \widehat{\otimes}_{R_{-n} \widehat{\otimes}_W R_0, \gamma} \mathrm{H}^j(M^\bullet) \cong R \widehat{\otimes}_{R_j, \gamma_j} X_j$$

for $j = -n$ and $j = 0$. Because R_{-n} (resp. R_0) is the universal deformation ring of U_{-n} (resp. U_0), in the sense of Mazur, it follows that γ_{-n} (resp. γ_0) is the unique morphism in $\widehat{\mathcal{C}}$ so that $\mathrm{H}^{-n}(N^\bullet) \cong R \widehat{\otimes}_{R_{-n}, \gamma_{-n}} X_{-n}$ (resp. $\mathrm{H}^0(N^\bullet) \cong R \widehat{\otimes}_{R_0, \gamma_0} X_0$). This implies that γ is unique, and thus R^{fl} is universal. □

PROPOSITION 9.8. — *With the hypotheses of Proposition 9.6, suppose in addition that U_{-n}, U_0 have universal deformation rings R_{-n}, R_0 and universal deformations X_{-n}, X_0, in the sense of Mazur, such that (9.8) is satisfied. Further assume that G has cohomological dimension $n + 1$. Then there exists a proflat quasi-lift $(M^\bullet, \phi)$ of $V^\bullet$ over $R_{-n} \widehat{\otimes}_W R_0$ satisfying (9.9).*

Proof. — Let $R = R_{-n} \widehat{\otimes}_W R_0$, and define $S_{-n} = R \widehat{\otimes}_{R_{-n}} X_{-n}$ and $S_0 = R \widehat{\otimes}_{R_0} X_0$. We want to find an element $\gamma \in \mathrm{Ext}^{n+1}_{[[RG]]}(S_0, S_{-n})$ so that $k \widehat{\otimes}_R \gamma = \beta$ where

$$\beta \in \mathrm{Ext}^{n+1}_{[[kG]]}(U_0, U_{-n}) = \mathrm{H}^{n+1}\big(G, \mathrm{Hom}_k(U_0, U_{-n})\big)$$

is defined by the triangle in (9.5). Since X_{-n} (resp. X_0) is an abstractly free R_{-n}-module (resp. R_0-module) of finite rank, it follows that S_{-n} and S_0 are abstractly free R-modules of finite rank. Therefore, a spectral sequence shows that

$$\mathrm{Ext}^{n+1}_{[[RG]]}(S_0, S_{-n}) = \mathrm{H}^{n+1}\big(G, \mathrm{Hom}_R(S_0, S_{-n})\big).$$

The module $\mathrm{Hom}_R(S_0, S_{-n})$ is an abstractly free R-module of finite rank with a certain G-action. We now consider the short exact sequence

$$0 \to C \xrightarrow{\sigma} \mathrm{Hom}_R(S_0, S_{-n}) \xrightarrow{\tau} \mathrm{Hom}_k(U_0, U_{-n}) \to 0$$

with corresponding long exact group cohomology sequence

$$\cdots \to \mathrm{H}^{n+1}\big(G, \mathrm{Hom}_R(S_0, S_{-n})\big) \xrightarrow{\tau_*} \mathrm{H}^{n+1}\big(G, \mathrm{Hom}_k(U_0, U_{-n})\big)$$
$$\longrightarrow \mathrm{H}^{n+2}(G, C) \to \cdots.$$

Since G has cohomological dimension $n + 1$, $\mathrm{H}^{n+2}(G, C) = 0$ and τ_* is surjective. Since the map τ_* can be identified with the tensor product map $k \widehat{\otimes}_R -$, it follows that there exists an element $\gamma \in \mathrm{Ext}^{n+1}_{[[RG]]}(S_0, S_{-n})$ with $k \widehat{\otimes}_R \gamma = \beta$. Now γ defines an exact sequence of $[[RG]]$-modules

$$0 \to S_{-n} \longrightarrow M^{-n} \longrightarrow M^{-n+1} \to \cdots \to M^{-1} \to M^0 \longrightarrow S_0 \to 0$$

which defines an $(n + 1)$-term complex

$$M^\bullet : M^{-n} \longrightarrow M^{-n+1} \to \cdots \to M^{-1} \longrightarrow M^0.$$

Since $k \widehat{\otimes}_R \gamma = \beta$, it follows that $M^\bullet$ defines a proflat quasi-lift of $V^\bullet$ over R with cohomology groups S_{-n} and S_0 in dimensions $-n$ and 0. Hence (9.9) is satisfied. □

10. Deforming group cohomology elements.

In this section we consider deformations of elements of group cohomologies $\mathrm{H}^n(G, M)$ where G is a profinite group having finite pseudo-compact cohomology and M is a discrete $[[kG]]$-module of finite k-dimension. This is closely related to the previous section about two-term complexes. Suppose

$$\beta \in \mathrm{H}^n(G, M) = \mathrm{Ext}^n_{[[kG]]}(k, M) = \mathrm{Hom}_{D^-([[kG]])}\big(k, T^n(M)\big)$$

where the last equation holds when we identify the $[[kG]]$-modules k and M with the corresponding complexes concentrated in dimension 0.

DEFINITION 10.1. — A *quasi-lift* of $\beta \in \mathrm{H}^n(G, M)$ is defined to be a quasi-lift of the mapping cone $C(\beta)^\bullet$ (as defined in Definition 9.2). A *deformation* of β is defined to be a deformation of $C(\beta)^\bullet$. *Proflat quasi-lifts* and *proflat deformations* of β are defined accordingly.

Remark 10.2. — The element $\beta \in \mathrm{H}^n(G, M)$ defines an n-extension of k by M:

$$0 \to M \longrightarrow P^{-n+1} \longrightarrow P^{-n+2} \to \cdots \to P^0 \longrightarrow k \to 0.$$

The corresponding triangle has the form

$$T^{n-1}(M) \longrightarrow V^\bullet \longrightarrow k \stackrel{\beta}{\longrightarrow} T^n(M)$$

where $V^\bullet : \cdots \to 0 \to P^{-n+1} \to P^{-n+2} \to \cdots \to P^0 \to 0 \to \cdots$. It follows that $C(\beta)^\bullet$ is isomorphic to $T(V^\bullet)$ in $D^-([[kG]])$. Hence a quasi-lift of β is the same as a quasi-lift of $T(V^\bullet)$. In particular, quasi-lifts of $\beta \in \mathrm{H}^1(G, M)$ correspond to quasi-lifts of the module P^0.

For the remainder of this section, we consider the following example.

Let $\ell > 2$ be a rational prime and let $G = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Define $M = \{\pm 1\}$, which forces the field k to be $k = \mathbb{Z}/2$, and hence $M = k$ with trivial G-action. Let $W = \mathbb{Z}_2$. We want to discuss deformations of nontrivial elements of $\mathrm{H}^2(G, M)$. Because of the Kummer sequence

$$1 \to \{\pm 1\} \longrightarrow \overline{\mathbb{Q}}_\ell^* \stackrel{\cdot 2}{\longrightarrow} \overline{\mathbb{Q}}_\ell^* \to 1$$

we obtain that $\mathrm{H}^2(G, M) = \mathbb{Z}/2$ has exactly one nontrivial element β.

LEMMA 10.3. — *The mapping cone $C(\beta)^\bullet$ is isomorphic to $T(V^\bullet)$ where*

$$V^\bullet : \cdots \to 0 \longrightarrow [kG_b] \stackrel{\delta}{\longrightarrow} [kG_a] \to 0 \to \cdots$$

and $a = \ell$, b is an element of $\mathbb{Z}_\ell^*$ which is not a square $\mathrm{mod}\,\ell$, $G_a = \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt{a})/\mathbb{Q}_\ell)$, $G_b = \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt{b})/\mathbb{Q}_\ell)$ and δ is the augmentation map of $[kG_b]$ composed with multiplication by $1 + s$ when $G_a = \{1, s\}$.

Proof. — For $x = a, b$, the extension

$$0 \to k \longrightarrow [kG_x] \longrightarrow k \to 0$$

in $\mathrm{Ext}^1_{[[kG]]}(k, k) = \mathrm{H}^1(G, k) \cong \mathrm{H}^1(G, \{\pm 1\})$ resulting from the augmentation map $[kG_x] \to k$ defines the element h_x of $\mathrm{H}^1(G, \{\pm 1\}) = \mathrm{Hom}(G, \{\pm 1\})$ associated to the canonical surjection $G \to G_x \cong \{\pm 1\}$. The cup product $h_a \cup h_b$ is equal to the Hilbert symbol $(a, b) \in \mathrm{H}^2(G, \{\pm 1\})$, which is β when a and b are chosen as in Lemma 10.3. This lemma now follows from the fact that $V^\bullet$ realizes the 2-extension $h_a \cup h_b$. $\square$

Remark 10.4. — Since $V^\bullet$ does not satisfy condition (i) of Proposition 9.3, $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet, V^\bullet)$ is not isomorphic to k. Thus we cannot deduce from this Proposition that $V^\bullet$ has a universal (proflat) deformation ring.

LEMMA 10.5. — *The tangent spaces of $\widehat{F} = \widehat{F}_{V^\bullet}$ and of $\widehat{F}^{\mathrm{fl}} = \widehat{F}^{\mathrm{fl}}_{V^\bullet}$ are both 4-dimensional over k.*

Proof. — We use the following diagram with exact rows and columns obtained from long exact Hom sequences associated to the triangle $T(k) \longrightarrow V^\bullet \longrightarrow k \stackrel{\beta}{\longrightarrow} T^2(k)$:

$$(10.1)$$

We have the following equalities:

$$\mathrm{H}^0(G, k) = k, \quad \mathrm{H}^1(G, k) = k \oplus k, \quad \mathrm{H}^2(G, k) = k, \quad \mathrm{H}^3(G, k) = 0.$$

In the fifth column of the diagram (10.1), we have $\mathrm{Hom}(T(k), k) = \mathrm{Hom}(k, T^{-1}(k)) = 0$, $\mathrm{Ext}^1(k, k) = \mathrm{H}^1(G, k) = k \oplus k$, $\mathrm{Ext}^1(T(k), k) = \mathrm{Hom}(T(k), T(k)) = \mathrm{H}^0(G, k) = k$ and $\mathrm{Ext}^2(k, k) = \mathrm{H}^2(G, k) = k$. It follows that the vertical map in the fifth column $\mathrm{Ext}^1(T(k), k) \to \mathrm{Ext}^2(k, k)$, and thus the map $\mathrm{Ext}^1(k, k) \to \mathrm{Ext}^1(V^\bullet, k)$ are isomorphisms, and $\mathrm{Ext}^1(V^\bullet, k) = k \oplus k$.

Since $\mathrm{Ext}^2(k, T(k)) = \mathrm{H}^3(G, k) = 0$, the horizontal map in the third row $\mathrm{Ext}^1(V^\bullet, k) \to \mathrm{Ext}^2(V^\bullet, T(k))$ is the zero map.

In the third column of the diagram (10.1), we have $\mathrm{Hom}(T(k), T(k)) = \mathrm{H}^0(G, k) = k$, $\mathrm{Ext}^1(k, T(k)) = \mathrm{H}^2(G, k) = k$, $\mathrm{Ext}^1(T(k), T(k)) = \mathrm{H}^1(G, k) = k \oplus k$ and $\mathrm{Ext}^2(k, T(k)) = \mathrm{H}^3(G, k) = 0$. It follows that the vertical map in the third column $\mathrm{Hom}(T(k), T(k)) \to \mathrm{Ext}^1(k, T(k))$, and thus the map $\mathrm{Ext}^1(V^\bullet, T(k)) \to \mathrm{Ext}^1(T(k), T(k))$ are isomorphisms, and $\mathrm{Ext}^1(V^\bullet, T(k)) = k \oplus k$.

Because $\mathrm{Ext}^{-1}(T(k), k) = 0$, $\mathrm{Hom}(k, k) = \mathrm{H}^0(G, k) = k$ and $\mathrm{Hom}(T(k), k) = 0$, the vertical map in the second column $\mathrm{Hom}(k, k) \to \mathrm{Hom}(V^\bullet, k)$ is an isomorphism, and $\mathrm{Hom}(V^\bullet, k) = k$.

This implies that the horizontal map $\mathrm{Hom}(V^\bullet, V^\bullet) \to \mathrm{Hom}(V^\bullet, k)$ in the third row is surjective, and therefore the horizontal map in the third row $\mathrm{Hom}(V^\bullet, k) \to \mathrm{Ext}^1(V^\bullet, T(k))$ is the zero map. Taken altogether, this implies that $\mathrm{Ext}^1(V^\bullet, V^\bullet) = \mathrm{Ext}^1(V^\bullet, T(k)) \oplus \mathrm{Ext}^1(V^\bullet, k)$ which means that the tangent space of $\widehat{F}$ is 4-dimensional over k.

To compute the tangent space of $\widehat{F}^{\mathrm{fl}}$ we use Proposition 9.6. Since the map $\mathrm{Ext}^1(T(k), k) \to \mathrm{Ext}^2(k, k)$ is an isomorphism, it is especially injective, which implies by Proposition 9.6(iii) that $t_F = t_{F^{\mathrm{fl}}}$. This completes the proof of Lemma 10.5. $\square$

THEOREM 10.6. — *Let* $W = \mathbb{Z}_2$. *The versal proflat deformation ring of* $V^\bullet$ *is universal and isomorphic to the completed tensor product* $[[WG^{\mathrm{ab},2}]] \widehat{\otimes}_W [[WG^{\mathrm{ab},2}]]$ *where* $G^{\mathrm{ab},2}$ *denotes the abelianized 2-completion of* G.

Proof. — Let $U_{-1} = \mathrm{H}^{-1}(V^\bullet)$ and $U_0 = \mathrm{H}^0(V^\bullet)$. Since $U_{-1} = k$ (resp. $U_0 = k$) with trivial G-action, it has a universal deformation ring, in the sense of Mazur, which is $R_{-1} \cong [[WG^{\mathrm{ab},2}]]$ (resp. $R_0 \cong [[WG^{\mathrm{ab},2}]]$) by [19, §1.4]. It follows that

$$\dim_k \mathrm{Ext}^1_{[[kG]]}(U_{-1}, U_{-1}) = 2 = \dim_k \mathrm{Ext}^1_{[[kG]]}(U_0, U_0).$$

By Lemma 10.5, condition (9.8) is satisfied. Since G has cohomological dimension 2, Theorem 10.6 follows from Propositions 9.7 and 9.8. $\square$

11. Completely split complexes.

Throughout this section, we assume that $V^\bullet$ is isomorphic in $D^-([[kG]])$ to a complex whose boundary maps are trivial. Thus $V^\bullet$ is isomorphic in $D^-([[kG]])$ to the direct sum $\bigoplus_i T^{-i}(\mathrm{H}^i(V^\bullet))$, where there are only finitely many non-zero terms in this sum and all terms have finite k-dimension by Hypothesis 1. Without loss of generality, we can assume that all the boundary maps of $V^\bullet$ are trivial, so $\mathrm{H}^i(V^\bullet) = V^i$ for all i.

DEFINITION 11.1. — A *split quasi-lift* of $V^\bullet$ over an object R of $\widehat{\mathcal{C}}$ is a proflat quasi-lift $(M^\bullet, \phi)$ which is isomorphic in $D^-([[RG]])$ to a complex whose boundary maps are trivial. A *split deformation* is the isomorphism class of a split quasi-lift. Let $\widehat{F}^{\mathrm{sp}} = \widehat{F}^{\mathrm{sp}}_{V^\bullet} \colon \widehat{\mathcal{C}} \to \mathrm{Sets}$ be the functor which sends each object R of $\widehat{\mathcal{C}}$ to the set $\widehat{F}^{\mathrm{sp}}(R)$ of all split deformations of $V^\bullet$ over R. Let $F^{\mathrm{sp}} = F^{\mathrm{sp}}_{V^\bullet}$ be the restriction of $\widehat{F}^{\mathrm{sp}}$ to $\mathcal{C}$.

LEMMA 11.2. — *The functor* $\widehat{F}^{\mathrm{sp}}$ (*resp.* F^{sp}) *is isomorphic to the product of the functors on* $\widehat{\mathcal{C}}$ (*resp.* $\mathcal{C}$) *associated to deformations, in the sense of Mazur, of the non-zero cohomology groups of* $V^\bullet$ *considered as* $[[kG]]$-*modules. Moreover, the functor* $\widehat{F}^{\mathrm{sp}}$ (*resp.* F^{sp}) *is naturally isomorphic to the functor* $\widehat{F}^{\mathrm{fl}}$ (*resp.* F^{fl}).

Proof. — If R is an object of $\widehat{\mathcal{C}}$ and M is an $[[RG]]$-module, we will also regard M as a complex concentrated in dimension 0. A split quasi-lift $M^\bullet$ of $V^\bullet$ over R is isomorphic in $D^-([[RG]])$ to the direct sum $\bigoplus_i T^{-i}(\mathrm{H}^i(M^\bullet))$, where the sum is over those i for which $\mathrm{H}^i(V^\bullet) = V^i \neq \{0\}$. Since the $\mathrm{H}^i(M^\bullet)$ are topologically free pseudocompact R-modules, $k \widehat{\otimes}_R {}^{\boldsymbol{L}} M^\bullet$ is isomorphic in $D^-([[kG]])$ to $\bigoplus_i T^{-i}(k \widehat{\otimes}_R \mathrm{H}^i(M^\bullet))$. By Lemma 2.11, $\mathrm{H}^i(M^\bullet)$ is an abstractly free R-module of rank $\dim_k V^i$. Hence $\mathrm{H}^i(M^\bullet)$ is a lift over R of the kG-module V^i in the sense of Mazur. This implies that $\widehat{F}^{\mathrm{sp}}$ is isomorphic to the product of the functors on $\widehat{\mathcal{C}}$ associated to deformations, in the sense of Mazur, of the non-zero cohomology groups of $V^\bullet$ considered as $[[kG]]$-modules.

Suppose now that $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$ and $(M^\bullet, \phi)$ is a proflat quasi-lift of $V^\bullet$ over R. By Lemma 2.9, we may assume that the terms of $M^\bullet$ are topologically free pseudocompact R-modules. If we denote the i-th boundary map of $M^\bullet$ by δ^i, then $k \widehat{\otimes}_R M^\bullet \cong V^\bullet$ in $D^-([[kG]])$ implies

$$(11.1) \qquad\qquad k \widehat{\otimes}_R \mathrm{Image}(\delta^i) = 0$$

for all i. As in (2.1), it follows that $\mathrm{Image}(\delta^i)$ is a topologically free pseudocompact R-module for all i. Thus (11.1) implies that $\mathrm{Image}(\delta^i) = 0$ for all i. Hence $M^\bullet$ is isomorphic in $D^-([[RG]])$ to a complex whose boundary maps are trivial, and defines therefore a split quasi-lift of $V^\bullet$ over R. This completes the proof of Lemma 11.2. $\square$

PROPOSITION 11.3. — *Suppose as before that* G *has finite pseudocompact cohomology.*

(i) *The versal split deformation ring* $R^{\mathrm{sp}}(G, V^\bullet)$ *associated to the split deformation functor* $\widehat{F}^{\mathrm{sp}}$ *is the tensor product* $\widehat{\bigotimes}_i R(G, V^i)$ *over* W *of the versal deformation rings, in the sense of Mazur, of the non-zero cohomology groups of* $V^\bullet$. *A versal split deformation is given by the direct sum*

$$U^{\mathrm{sp}}(G, V^\bullet) = \bigoplus_i T^{-i}\big(R^{\mathrm{sp}}(G, V^\bullet) \widehat{\otimes}_{R(G, V^i)} U(G, V^i)\big)$$

where i *runs over those integers for which* $\mathrm{H}^i(V^\bullet) \neq \{0\}$.

(ii) *The natural map on tangent spaces* $\tau\colon F^{\mathrm{sp}}(k[\varepsilon]) \to F(k[\varepsilon])$ *may be identified with the natural inclusion*

$$\iota\colon \bigoplus_i \mathrm{Ext}^1_{[[kG]]}(V^i,V^i) \longrightarrow \mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet,V^\bullet) = \bigoplus_{i,j} \mathrm{Ext}^{1+i-j}_{[[kG]]}(V^i,V^j).$$

We have a non-canonical surjective continuous W-algebra homomorphism $f_{\mathrm{sp}}\colon R(G,V^\bullet) \to R^{\mathrm{sp}}(G,V^\bullet).$

(iii) *Suppose that each $R(G,V^i)$ is isomorphic to a power series algebra over W on a finite number of commuting indeterminates. Then f_{sp} is an isomorphism if and only if τ is an isomorphism.*

(iv) *If the versal deformation ring $R(G,V^i)$ is universal for all i, then $R^{\mathrm{sp}}(G,V^\bullet)$ is a universal split deformation ring. If in addition f_{sp} is an isomorphism, then $R(G,V^\bullet)$ is a universal deformation ring for $V^\bullet$.*

Proof. — By Lemma 11.2, $\widehat{F}^{\mathrm{sp}}$ has a versal deformation ring $R^{\mathrm{sp}}(G,V^\bullet)$. The fact that $R^{\mathrm{sp}}(G,V^\bullet)$ is the completed tensor product $\widehat{\bigotimes}_i R(G,V^i)$ and the complex $U^{\mathrm{sp}}(G,V^\bullet)$ defined in part (i) is a versal split deformation for $V^\bullet$ results from the fact that $\widehat{\bigotimes}_i R(G,V^i)$ is the coproduct of the $R(G,V^i)$ over W in the category $\widehat{\mathcal{C}}$. The identification of τ with ι follows from Lemma 6.1 and the fact that all the boundary maps of $V^\bullet$ have been assumed to be trivial. This shows τ is injective, so f_{sp} is surjective. We have now shown parts (i) and (ii). Part (iii) follows from the fact that $R(G,V^\bullet)$ is a quotient of a power series algebra over W on $\dim_k \mathrm{Ext}^1_{D^-([[kG]])}(V^\bullet,V^\bullet)$ commuting indeterminates. Part (iv) is clear because, if all the $R(G,V^i)$ are universal, and R is an object in $\widehat{\mathcal{C}}$, then there is a unique continuous W-algebra homomorphism from $R^{\mathrm{sp}}(G,V^\bullet) = \widehat{\bigotimes}_i R(G,V^i)$ to R associated with a collection of such homomorphisms from $R(G,V^i)$ to R. □

Remark 11.4. — When $V^\bullet$ is completely split, the only way in which $\mathrm{Hom}_{D^-([[kG]])}(V^\bullet,V^\bullet)$ can be isomorphic to k is when $V^\bullet$ has just one non-zero term. Part (iv) of Proposition 11.3 thus gives another situation than the one discussed in Theorem 2.14 in which $V^\bullet$ has a universal deformation ring.

12. The hypercohomology of abelian varieties.

In this section we discuss some sufficient conditions for the hypercohomology of an twisted constant constructible sheaf on an abelian variety to be completely split, in the sense that it is isomorphic in the derived category to a complex having trivial boundary maps. The method we use is due to Lieberman (cf. [11, Rem. 2.19]).

Let X be an abelian variety over $\mathbb{Q}$ of dimension d, and let $\overline{X} = \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} X$. Let p be a prime. Let k be a finite field of characteristic p with trivial action by $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We will suppose that $\mathcal{F}$ is a locally constant constructible sheaf o k-vector spaces on X. Denote by $\mathcal{F}(n)$ the n-th Tate twist $\mathcal{F}(\mu_p^{\otimes n})$ of $\mathcal{F}$. By [21, Cor. VI.2.8 and Thm. VI.1.1], the cohomology groups $\mathrm{H}^i(\overline{X},\mathcal{F})$ are finite continuous $[[kG_{\mathbb{Q}}]]$-modules which are zero unless $0 \le i \le 2d$. Thus the hypercohomology $\mathrm{H}^\bullet(\overline{X},\mathcal{F})$ defines an element in $D^-([[kG_{\mathbb{Q}}]])$ which has only finitely many non-zero cohomology groups, all of which are finite.

THEOREM 12.1. — *If $p > 2d$, the complex $\mathrm{H}^\bullet(\overline{X},\mathcal{F})$ is completely split in $D^-([[kG_{\mathbb{Q}}]])$. If there is a non-degenerate k-bilinear pairing of constructible sheaves $\mathcal{F} \times \mathcal{F} \to k(d)$, then the same conclusion holds provided $p > 2d - 2$.*

We need two Lemmas, the first of which is a consequence of [11, Theorem I.II].

LEMMA 12.2. — *Suppose $V^\bullet$ is a complex in $D^-([[kG]])$ as in Hypothesis 1. Let c be the k-linear map*

$$c\colon \mathrm{End}_{D^-([[kG]])}(V^\bullet) = \mathrm{Hom}_{D^-([[kG]])}(V^\bullet,V^\bullet)$$
$$\longrightarrow R = \bigoplus_i \mathrm{End}_{[[kG]]}\big(\mathrm{H}^i(V^\bullet)\big)$$

induced by taking the action of endomorphisms on cohomology. The complex $V^\bullet$ is completely split if and only for all j, the image of c contains the element β_j of R whose projection to $\mathrm{End}_{[[kG]]}(\mathrm{H}^i(V^\bullet))$ is the identity map (resp. zero) if $i = j$ (resp. $i \ne j$).

LEMMA 12.3. — *For each integer $n \ne 0$ let $m_n\colon \overline{X} \to \overline{X}$ be the map given by multiplication by n. For all integers i, the induced homomorphism $m^*_{n,i}\colon \mathrm{H}^i(\overline{X},\mathcal{F}) \to \mathrm{H}^i(\overline{X},\mathcal{F})$ is multiplication by n^i.*

Proof. — From [21, Cor. VI.2.6 and Thm. III.3.12], [15, p. 302] and the Universal Coefficient Theorem we have an isomorphism

$$\mathrm{H}^i(\overline{X},\mathcal{F}) = \big(\textstyle\bigwedge^i_{\mathbb{Z}_p} \mathrm{H}^1(\overline{X},\mathbb{Z}_p)\big) \otimes_{\mathbb{Z}_p} M$$

where $M = \mathcal{F}(\overline{X})$ is a finite dimensional k-vector space. Since $\mathrm{H}^1(\overline{X},\mathbb{Z}_p)) = \mathrm{Hom}_{\mathrm{cont}}(T_p(\overline{X}),\mathbb{Z}_p)$ when $T_p(\overline{X})$ is the p-adic Tate module of $\overline{X}$, and

multiplication by n on $\overline{X}$ induces multiplication by n on $T_p(\overline{X})$, Lemma 12.3 follows.　□

Proof of Theorem 12.1. — Set $V^\bullet = \mathrm{H}^\bullet(\overline{X}, \mathcal{F})$, and consider the endomorphisms $m_n^\bullet$ in $\mathrm{End}_{D^-([[kG_\mathbb{Q}]])}(V^\bullet)$ which are induced by the multiplication by n map $m_n : \overline{X} \to \overline{X}$ for $n = 1, \ldots, 2d+1$. Let

$$c : \mathrm{End}_{D^-([[kG_\mathbb{Q}]])}(V^\bullet) \longrightarrow R = \bigoplus_i \mathrm{End}_{[[kG_\mathbb{Q}]]}\big(\mathrm{H}^i(V^\bullet)\big)$$

be the homomorphism of Lemma 12.2. Lemma 12.3 shows that the image of $m_n^\bullet$ under the homomorphism c is the element of R whose i-th component is multiplication by n^i, where i runs from 0 to $2d$. Thus on viewing elements of R as column matrices, we see that $\{c(m_n^\bullet)\}_{n=1}^{2d+1}$ defines a square matrix of size $(2d+1)$ whose (i,n) entry is the endomorphism of $\mathrm{H}^i(V^\bullet)$ given by multiplication by n^i. The matrix of integers $\{n^i\}_{i,n}$ as i ranges over $0 \le i \le 2d$ and n ranges over $1 \le n \le 2d+1$ has a Vandermonde determinant. If $p > 2d$, then this Vandermonde determinant is non-zero modulo p because $i - j \not\equiv 0 \bmod p$ if $1 \le i \ne j \le 2d+1$. Therefore, since $pR = \{0\}$, we conclude that the additive subgroup of R generated by $\{c(m_n^\bullet)\}_{n=1}^{2d+1}$ contains the β_j for $0 \le j \le 2d$ described in Lemma 12.2. Thus the latter lemma implies $V^\bullet$ is completely split if $p > 2d$.

To complete the proof of Theorem 12.1, we now suppose that there is a non-degenerate pairing $\mathcal{F} \times \mathcal{F} \to k(d)$ of constructible sheaves on X, where $k(d)$ is the d-th Tate twist of the constant sheaf k. This pairing produces an isomorphism of étale sheaves

$$(12.1) \qquad \mathcal{F} \longrightarrow \check{\mathcal{F}}(d)$$

where $\check{\mathcal{F}} = \mathrm{Hom}_k(\mathcal{F}, k)$. We now take advantage of the fact that because X is an abelian variety, it has at least one $\mathbb{Q}$-rational point, namely the origin Q. Define $\overline{Q} = \overline{\mathbb{Q}} \otimes_\mathbb{Q} Q$, and let $\iota : \overline{Q} \to \overline{X}$ be the inclusion map. Restriction from $\overline{X}$ to $\overline{Q}$ induces a morphism $\mathrm{H}^\bullet(\overline{X}, \mathcal{F}) = V^\bullet \to \mathrm{H}^\bullet(\overline{Q}, \mathcal{F})$ which gives an isomorphism $\mathrm{H}^0(\overline{X}, \mathcal{F}) = \mathrm{H}^0(\overline{Q}, \mathcal{F}) = \mathcal{F}(\overline{X})$ in degree 0, and the zero map on cohomology in other degrees. This morphism is the inverse in $D^-([[kG_\mathbb{Q}]])$ of the morphism $\mathcal{F}(\overline{X}) \to \mathrm{H}^\bullet(\overline{X}, \mathcal{F}) = V^\bullet$ which results from truncating the terms of $V^\bullet$ in negative degrees and replacing V^0 by its image in V^1 under the boundary map. The composition of these morphisms is thus an endomorphism τ_0 of $V^\bullet$ which is the identity map on $\mathrm{H}^0(V^\bullet)$ and zero on all other cohomology groups.

The Étale Duality Theorem (cf. [21, Thm. VI.11.10] and [16, XVIII]) shows that there is an isomorphism in $D^-([[kG_\mathbb{Q}]])$ between $V^\bullet = \mathrm{H}^\bullet(\overline{X}, \mathcal{F})$ and $\mathrm{Hom}_k(T^{2d}(\mathrm{H}^\bullet(\overline{X}, \check{\mathcal{F}}(d))), k)$. Using the isomorphism in (12.1), we thus obtain an isomorphism

$$(12.2) \qquad \lambda : V^\bullet \longrightarrow \mathrm{Hom}_k\big(T^{2d}(V^\bullet), k\big) \quad \text{in } D^-\big([[kG_\mathbb{Q}]]\big).$$

The endomorphism τ_0 of $V^\bullet$ defines a dual endomorphism of

$$\mathrm{Hom}_k\big(T^{2d}(V^\bullet), k\big) = T^{-2d}\big(\mathrm{Hom}_k(V^\bullet, k)\big),$$

and thus via λ an endomorphism τ_{2d} of $V^\bullet$. Consider the effect of τ_{2d} on cohomology. The isomorphism λ identifies $\mathrm{H}^i(V^\bullet)$ with $\mathrm{Hom}_k(\mathrm{H}^{2d-i}(V^\bullet), k)$. Hence τ_{2d} induces the identity map on $\mathrm{H}^{2d}(V^\bullet)$, and the zero-map on $\mathrm{H}^i(V^\bullet)$ for all other i. We now define $\mathcal{E}$ to be the set of endomorphisms of $V^\bullet$ given by τ_0, τ_{2d} and $\{m_n^\bullet\}_{n=1}^{2d-1}$. The action of this set of endomorphisms on the cohomology of $V^\bullet$ is represented by a $(2d+1) \times (2d+1)$ matrix $A = (A_{i,j})_{0 \le i,j \le 2d}$ of scalars. We have $A_{0,0} = A_{2d,2d} = 1$, $A_{i,0} = A_{i',2d} = 0$ for $i \ne 0$ and $i' \ne 2d$, and $A_{i,j} = j^i$ for all i and all $1 \le j \le 2d-1$. The determinant of A is thus the Vandermonde determinant associated to the integers in the interval $[1, 2d-1]$. If $p > 2d-2$ and p is prime, this determinant is non-zero mod p. We thus find as before that if $p > 2d-2$, then the additive group of endomorphisms of $V^\bullet$ generated by $\mathcal{E}$ is large enough to satisfy the sufficient condition in Lemma 12.2 for $V^\bullet$ to be completely split. This completes the proof of Theorem 12.1.　□

COROLLARY 12.4. — *Let Q be the origin of X. Define $U = X - \{Q\}$, $\overline{U} = \overline{\mathbb{Q}} \otimes_\mathbb{Q} U$ and $\overline{Q} = \overline{\mathbb{Q}} \otimes_\mathbb{Q} Q$. The compact étale hypercohomology $\mathrm{H}_c^\bullet(\overline{U}, \mathcal{F})$ is an element in $D^-([[kG_\mathbb{Q}]])$. This element is completely split if $p > 2d$. The same is true for $p > 2d-2$ provided there is a non-degenerate k-bilinear pairing of constructible sheaves $\mathcal{F} \times \mathcal{F} \to k(d)$.*

Proof. — By [21, Remark III.1.30], there is a triangle in $D^-([[kG_\mathbb{Q}]])$

$$(12.3) \qquad \mathrm{H}_c^\bullet(\overline{U}, \mathcal{F}) \longrightarrow \mathrm{H}^\bullet(\overline{X}, \mathcal{F}) \overset{\alpha}{\longrightarrow} \mathrm{H}_c^\bullet(\overline{Q}, \mathcal{F}) \longrightarrow T\big(\mathrm{H}_c^\bullet(\overline{U}, \mathcal{F})\big).$$

The complex $\mathrm{H}_c^\bullet(\overline{Q}, \mathcal{F}) = \mathrm{H}^\bullet(\overline{Q}, \mathcal{F})$ has only one non-zero cohomology group, which is in dimension 0 and equal to $\mathcal{F}(\overline{\mathbb{Q}}) = \mathrm{H}^0(\overline{X}, \mathcal{F})$. Since $T(\mathrm{H}_c^\bullet(\overline{U}, \mathcal{F}))$ is isomorphic to the mapping cone of α, Corollary 12.4 follows from Theorem 12.1.　□

COROLLARY 12.5. — *With the notations of Corollary 12.4, suppose $d = 1$, so that X is an elliptic curve. For all primes p, the complexes $\mathrm{H}^\bullet(\overline{X}, \mu_p)$ and $\mathrm{H}_c^\bullet(\overline{U}, \mu_p)$ are completely split in $D^-([[kG_\mathbb{Q}]])$.*

Proof. — In view of Theorem 12.1, all we need to observe is that multiplication defines a non-degenerate pairing $\mu_p \times \mu_p \to \mu_p(1) = \mu_p^{\otimes 2}$. (This pairing is in fact only needed when $p = 2$.) □

13. The hypercohomology of affine elliptic curves.

In this section we assume the notations of §12, and we suppose that $k = \mathbb{Z}/p$, $W = \mathbb{Z}_p$, and $d = 1$. Thus X is an elliptic curve over $\mathbb{Q}$ with origin Q, $U = X - \{Q\}$, and $\overline{X}$, $\overline{U}$ and $\overline{Q}$ are the base change of X, U and Q from $\mathbb{Q}$ to $\overline{\mathbb{Q}}$. Let N be the conductor of X, and let S be a finite set of places of $\mathbb{Q}$ containing the places determined by prime numbers dividing pN together with the archimedean place of $\mathbb{Q}$. Define $\mathbb{Q}_S$ to be the maximal algebraic extension of $\mathbb{Q}$ which is unramified outside S, and let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ containing $\mathbb{Q}_S$. We set $G_S = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$.

LEMMA 13.1. — *We have the following $[[kG_{\mathbb{Q}}]]$-module isomorphisms:*

$$(13.1) \qquad \mathrm{H}^i_c(\overline{U},\mu_p) = \begin{cases} \mathrm{Pic}(\overline{X})[p] & \text{if } i = 1, \\ \mathbb{Z}/p & \text{if } i = 2, \\ 0 & \text{otherwise} \end{cases}$$

where $G_{\mathbb{Q}}$ acts trivially on $\mathbb{Z}/p$.

Proof. — This is clear from the long exact cohomology sequence associated to (12.3) and the calculation of $\mathrm{H}^i(\overline{X},\mu_p)$ in [21, p. 125 and Thm. V.2.1]. □

COROLLARY 13.2. — *There is a $G_{\mathbb{Q}}$-module isomorphism between $\mathrm{Pic}(\overline{X})[p]$ and $\overline{X}[p]$. The $G_{\mathbb{Q}}$-modules $\overline{X}[p]$ and $\mathbb{Z}/p$ are inflated from G_S-modules, and thus may be regarded themselves as G_S-modules. Define $V^\bullet$ to be the complex of $[[kG_S]]$-modules $(\mathbb{Z}/p) \bigoplus T(\overline{X}[p])$, so that $V^\bullet$ has non-zero terms only in dimensions -1 and 0. Then the inflation of $V^\bullet$ from G_S to $G_{\mathbb{Q}}$ is isomorphic to $T^2(\mathrm{H}^\bullet_c(\overline{U},\mu_p))$ in $D^-([[kG_{\mathbb{Q}}]])$.*

Proof. — The $G_{\mathbb{Q}}$-isomorphism of $\mathrm{Pic}(\overline{X})[p]$ and $\overline{X}[p]$ is a consequence of the fact that $\overline{X}$ is an elliptic curve. Since S contains p, and the field obtained by adjoining to $\mathbb{Q}$ the coordinates of $\overline{X}[p]$ is unramified outside p, we see that the action of $G_{\mathbb{Q}}$ on $\overline{X}[p]$ factors through G_S. Clearly the trivial action of $G_{\mathbb{Q}}$ on $\mathbb{Z}/p$ factors through G_S, so Corollary 13.2 follows from Lemma 13.1 and Corollary 12.4. □

Remark 13.3. — Suppose X has good reduction at $p > 2$. It is well known that the group

$$\mathrm{Ext}^1_{[[kG_S]]}\big(\mathbb{Z}/p, \overline{X}[p]\big) = \mathrm{H}^1\big(G_S, \overline{X}[p]\big)$$

is then nontrivial. One can check this in the following way. The Weil pairing shows $\overline{X}[p]$ is G_S-isomorphic to its dual $\overline{X}[p]^D = \mathrm{Hom}(\overline{X}[p], \mathbb{G}_m) = \mathrm{Hom}(\overline{X}[p], \mu_p)$. By applying [22, Thm. I.4.10] to the module $M = \overline{X}[p]$, we see that if $\mathrm{H}^1(G_S, \overline{X}[p]) = 0$ then $\mathrm{H}^1(\mathbb{Q}_p, \overline{X}[p]) = 0$. Consider the Kummer sequence

$$0 \to \overline{X}[p] \longrightarrow X(\overline{\mathbb{Q}}_p) \overset{p}{\longrightarrow} X(\overline{\mathbb{Q}}_p) \to 0.$$

This sequence gives a short exact sequence

$$0 \to X(\mathbb{Q}_p)/pX(\mathbb{Q}_p) \longrightarrow \mathrm{H}^1\big(\mathbb{Q}_p, \overline{X}[p]\big) \longrightarrow \mathrm{H}^1\big(\mathbb{Q}_p, X(\overline{\mathbb{Q}}_p)\big)[p] \to 0.$$

However, $X(\mathbb{Q}_p)$ contains an open subgroup of finite index isomorphic to $\mathbb{Z}_p$ since X is an elliptic curve over $\mathbb{Q}$ having good reduction at p. So $\mathrm{H}^1(\mathbb{Q}_p, \overline{X}[p]) \neq \{0\}$. Thus by Proposition 9.3, $\mathrm{Hom}_{D^-([[kG_S]])}(V^\bullet, V^\bullet)$ will not be one-dimensional over k. Thus we cannot apply Theorem 2.14 to conclude that $V^\bullet$ has a universal deformation ring, though it will always have a versal deformation ring $R(G_S, V^\bullet)$.

In the remainder of this section we will discuss some examples in which one can show that $R(G_S, V^\bullet)$ is in fact universal, using Proposition 11.3 (iv). Recall that we assume $W = \mathbb{Z}_p$.

PROPOSITION 13.4. — *Suppose that $\overline{X}[p]$ has the following properties:*

   (i) $\mathrm{Hom}_{G_S}(k, \overline{X}[p]) = 0 = \mathrm{H}^2(G_S, \overline{X}[p])$.

   (ii) *The maximal abelian pro-p quotient $G_S^{\mathrm{ab},p}$ of G_S is isomorphic to $\mathbb{Z}_p$.*

   (iii) *The versal deformation ring $R(G_S, \overline{X}[p])$ is isomorphic to a power series algebra over W on a finite number r of commuting indeterminates.*

Then the versal deformation ring $R(G_S, V^\bullet)$ is isomorphic to the versal split deformation ring $R^{\mathrm{sp}}(G_S, V^\bullet)$, which is a power series algebra over W on $r + 1$ indeterminates. If $R(G_S, \overline{X}[p])$ is a universal deformation ring, then so is $R(G_S, V^\bullet)$.

Proof. — By Corollary 13.2, $V^\bullet$ is a split complex whose only non-zero cohomology groups are $\mathrm{H}^0(V^\bullet) = k$ and $\mathrm{H}^{-1}(V^\bullet) = \overline{X}[p]$. Condition (i) of Proposition 13.4 shows that the map τ on tangent spaces appearing in (ii) of Proposition 11.3 is an isomorphism. By work of Mazur [19], the versal deformation ring $R(G_S, k)$ is universal, and isomorphic to $[[WG^{\mathrm{ab},p}]]$. Thus condition (ii) of Proposition 13.4 implies $R(G_S, k)$ is a power series over W in one variable. Condition (iii) of Proposition 13.4 shows that condition (iii) of Proposition 11.3 is satisfied, so the conclusions of Proposition 13.4 follow from Proposition 11.3. $\qquad\square$

Remark 13.5. — The statement that $\mathrm{Hom}_{G_S}(k, \overline{X}[p]) = 0$ is equivalent to the condition that $X[p] = 0$, i.e. $\overline{X}$ has no nontrivial p-torsion points defined over $\mathbb{Q}$. If $p > 2$, then by class field theory, $G_S^{\mathrm{ab},p}$ will be isomorphic to $\mathbb{Z}_p$ if and only if no finite place $v \in S$ has residue characteristic congruent to 1 mod p.

LEMMA 13.6. — *Suppose $p > 2$ and $\mathrm{Hom}_{G_S}(k, \overline{X}[p]) = 0$. Then $\mathrm{H}^2(G_S, \overline{X}[p]) = 0$ if and only if for each finite place $v \in S$, the group $X(\mathbb{Q}_v)[p]$ of p-torsion points of X which are rational over $\mathbb{Q}_v$ consists only of the origin Q of X.*

Proof. — By [22, p. 83] we have an exact sequence

$$(13.2) \quad 0 \to \mathrm{III}_S^2\big(\mathbb{Q}, \overline{X}[p]\big) \longrightarrow \mathrm{H}^2\big(G_S, \overline{X}[p]\big)$$
$$\longrightarrow \bigoplus_{v \in S} \mathrm{H}^2\big(\mathbb{Q}_v, \overline{X}[p]\big) \longrightarrow \mathrm{H}^0\big(G_S, (\overline{X}[p])^D\big)^* \to 0$$

where $(\overline{X}[p])^D = \mathrm{Hom}(X[p], \mu_p)$ is G_S-isomorphic to $\overline{X}[p]$ by the Weil pairing, $A^* = \mathrm{Hom}(A, \mathbb{Q}/Z)$ is the Pontryagin dual of a finite abelian group A, and $\mathrm{III}_S^2(\mathbb{Q}, \overline{X}[p])$ is the Shafarevich group defined on [22, p. 70]. By [22, Remark I.6.18], $\mathrm{III}_S^2(\mathbb{Q}, \overline{X}[p]) = 0$. Because $\mathrm{Hom}_{G_S}(k, \overline{X}[p]) = 0$, we have $\mathrm{H}^0(G_S, (\overline{X}[p])^D)^* = \mathrm{H}^0(G_S, \overline{X}[p])^* = 0$. Thus (13.2) shows $\mathrm{H}^2(G_S, \overline{X}[p]) = 0$ if and only if $\mathrm{H}^2(\mathbb{Q}_v, \overline{X}[p]) = 0$ for all $v \in S$. If v is archimedean, this is clear because p is odd. For finite v, we have by duality (cf. [22, p. 83]) that $\mathrm{H}^2(\mathbb{Q}_v, \overline{X}[p])$ has the same order as $\mathrm{H}^0(\mathbb{Q}_v, (\overline{X}[p])^D) = \mathrm{H}^0(\mathbb{Q}_v, \overline{X}[p]) = X(\mathbb{Q}_v)[p]$. This implies Lemma 13.6. $\qquad\square$

HYPOTHESIS 2. — *The elliptic curve X has complex multiplication by the ring of integers $\mathcal{O}$ in an imaginary quadratic field L of class number 1. The odd prime p splits in L into a product of distinct prime ideals p_1 and p_2 of $\mathcal{O}$, and X has good (ordinary) reduction at p.*

LEMMA 13.7. — *Write $\#X(\mathbb{Z}/p) = 1 - a_p + p$, so that $a_p \in \mathbb{Z}$ is the trace of Frobenius at p. If $p > 7$ and $a_p \not\equiv 1 \bmod p$, then $X(\mathbb{Q}_p)[p] = \{Q\}$.*

Proof. — Since $p\mathcal{O} = p_1 \cdot p_2$ and p_1 and p_2 are distinct prime ideals,

$$(13.3) \qquad\qquad \overline{X}[p] = \overline{X}[p_1] \oplus \overline{X}[p_2]$$

where $\overline{X}[p_i]$ denotes the p_i-torsion subgroup of $\overline{X}(\mathbb{Q}_S)$. Each of the groups $\overline{X}[p_1]$ and $\overline{X}[p_2]$ are cyclic of order p, and $\mathrm{Gal}((L\mathbb{Q}_S)/L) = \Gamma$ acts on $\overline{X}[p_i]$ via a one-dimensional $k = \mathbb{Z}/p$ valued character χ_i. The completions L_{p_i} are each isomorphic to $\mathbb{Q}_p$. Suppose $X(\mathbb{Q}_p)[p] = X(L_{p_1})[p]$ contains a point other than Q. This point is then fixed by the action of the decomposition group $\Gamma_{w_1} \subset \Gamma$ of a prime w_1 over p_1 in $L\mathbb{Q}_S$. Hence we conclude from (13.3) that one of χ_1 or χ_2 must be trivial on Γ_{w_1}.

By elliptic class field theory (cf. [26]), the fact that p is odd implies that the field $L(\overline{X}[p_1])$ obtained by adjoining to L the coordinates of the points of $\overline{X}[p_1]$ contains the ray class field $L(p_1)$ of L of conductor p_1. The ramification degree of $L(p_1)$ over L is $(p-1)/\mu$ where μ is the number of roots of unity in $\mathcal{O}$. Since $\mu \leq 6$, we conclude that if $p > 7$ then p_1 must ramify in $L(\overline{X}[p_1])$. Thus it is impossible that χ_1 is trivial on the decomposition group Γ_{w_1}.

Suppose now that χ_2 is trivial on Γ_{w_1}. This implies that the coordinates of the points of $\overline{X}[p_2]$ lie in $L_{p_1} = \mathbb{Q}_p$. The reduction map $X(L_{p_1}) \to X(\mathcal{O}/p_1) = X(\mathbb{Z}/p)$ is injective on p_2-torsion points, since p_1 and p_2 are distinct primes of $\mathcal{O}$. Thus there is a non-zero p_2-torsion point of $X(L_{p_1})$ which maps to a non-zero p-torsion point of $X(\mathcal{O}/p_1) = X(\mathbb{Z}/p)$. This proves that $X(\mathbb{Z}/p)$ has order divisible by p, which is equivalent to $a_p \equiv 1 \bmod p$. Hence the assumption $a_p \not\equiv 1 \bmod p$ implies $X(\mathbb{Q}_p)[p] = \{Q\}$. $\qquad\square$

LEMMA 13.8. — *Suppose v is a finite place in S different from the place defined by p. If $X(\mathbb{Q}_v)[p] \neq \{Q\}$, then there is a principal prime ideal over v which has a generator congruent to 1 mod p_1. In particular, if v is fixed, there are only finitely many rational primes p for which it is possible that $X(\mathbb{Q}_v)[p] \neq \{Q\}$.*

Proof. — Let t be a prime ideal of $\mathcal{O}$ over v, so t is principal because $\mathcal{O}$ has class number 1. Since $X(\mathbb{Q}_v)[p] \subset X(L_t)[p]$, we see from (13.3) that if $X(\mathbb{Q}_v)[p] \neq \{Q\}$ then one of the characters χ_1 or χ_2 used in the proof of Lemma 13.7 must be trivial on the decomposition group $\Gamma_w \subset \Gamma = \mathrm{Gal}((L\mathbb{Q}_S)/L)$ of a place w of $L\mathbb{Q}_S$ over t. By replacing t by

its conjugate by the nontrivial element of $\mathrm{Gal}(L/\mathbb{Q})$, if necessary, we can assume that χ_1 is trivial on Γ_w. This implies that t splits in the ray class field $L(p_1)$ of L of conductor p_1. This will be the case if and only if there is a generator for t which is congruent to 1 modulo p_1. If λ is such a generator, then $\mathrm{Norm}_{L/\mathbb{Q}}(\lambda - 1) \equiv 0 \bmod p$. Since there are finitely many generators for a principal ideal over v in $\mathcal{O}$, the possible p for which this can be true are bounded if v is fixed. $\qquad \square$

A *quadratic progression* is a set of integers of the form $\mathbb{Z} \cap \{f(x) : x \in \mathbb{Q}\}$ for some quadratic polynomial $f(x) \in \mathbb{Q}[x]$. The following result is proved in [6, Cor. 1.2 and Prop. 5.4] for $W = \mathbb{Z}_p$.

PROPOSITION 13.9 (Boston-Ullom). — *Suppose X is an elliptic curve of conductor N with complex multiplication by the ring of integers $\mathcal{O}$ in an imaginary quadratic field L of class number 1. There is a set T of rational primes consisting of a finite set together with some (possibly no) primes in quadratic progression such that if $p \notin T$ and p splits in L, the following are true when we let S be the set of places determined by the prime numbers dividing pN together with the archimedean place of $\mathbb{Q}$:*

(i) *$p > 7$ and X has good (ordinary) reduction at p.*

(ii) *$a_p \not\equiv 1 \bmod p$.*

(iii) *No finite place $v \in S$ has residue characteristic congruent to 1 mod p.*

(iv) *The versal deformation ring $R(G_S, \overline{X}[p])$ is universal and isomorphic to a power series algebra over W in three commuting indeterminates.*

We now have the following result from Proposition 13.4, Lemmas 13.7 and 13.8 and Proposition 13.9. Note that $\overline{X}[p]$ is nontrivial absolutely irreducible because p is an odd prime of good reduction for X.

THEOREM 13.10. — *Suppose X is an elliptic curve of conductor N with complex multiplication by the ring of integers $\mathcal{O}$ in an imaginary quadratic field L of class number 1. There is a set T' of rational primes consisting of a finite set together with some (possibly no) primes in quadratic progression such that the following is true:*

If $p \notin T'$, p splits in L, and S is the set of places of $\mathbb{Q}$ determined by the prime numbers dividing pN together with the archimedean place of $\mathbb{Q}$, then the versal deformation ring $R(G_S, V^\bullet)$ is universal and isomorphic to a power series algebra over W in four commuting indeterminates.

14. Appendix: Background.

In this section, we provide a few results from Milne [21] which we have restated to fit our situation. The proofs are similar to the proofs in [21]. Note that in [21, Lemma VI.8.17] (resp. [21, Lemma VI.8.18]), the condition "π is surjective on terms" (resp. "ψ is surjective on terms") is necessary in the statement.

LEMMA 14.1 (see [21, Lemma VI.8.17]). — *Let $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$, let G be a profinite group, and let $M^\bullet \xrightarrow{\phi} L^\bullet \xleftarrow{\pi} N^\bullet$ be morphisms in $C^-([[RG]])$ such that π is a quasi-isomorphism which is surjective on terms. If $M^\bullet$ is a complex of topologically free pseudocompact $[[RG]]$-modules, there exists a morphism $\psi : M^\bullet \to N^\bullet$ in $C^-([[RG]])$ such that $\pi\psi = \phi$.*

Remark 14.2. — Suppose $R, R_0 \in \mathrm{Ob}(\widehat{\mathcal{C}})$ so that R_0 is a quotient ring of R. We write $X \to X_0$, $\phi \to \phi_0$ for the functor $R_0 \widehat{\otimes}_R -$. Let G be a profinite group, and suppose M, N are topologically free pseudocompact $[[RG]]$-modules. Then every continuous $[[R_0G]]$-module homomorphism $\pi : M_0 \to N_0$ can be lifted to a continuous $[[RG]]$-module homomorphism $\phi : M \to N$ so that $\pi = \phi_0$.

LEMMA 14.3 (see [21, Sublemma VI.8.20]). — *Let $R, R_0 \in \mathrm{Ob}(\widehat{\mathcal{C}})$ so that R_0 is a quotient ring of R, and let G be a profinite group. As in Remark 14.2, we write $X \to X_0$, $\phi \to \phi_0$ for the functor $R_0 \widehat{\otimes}_R -$. Let $\phi : L^\bullet \to M^\bullet$ be a morphism in $C^-([[RG]])$ of complexes of topologically free pseudocompact $[[RG]]$-modules. Then any morphism $L_0^\bullet \to M_0^\bullet$ in $C^-([[R_0G]])$ that is homotopic to ϕ_0 is of the form ψ_0, where $\psi : L^\bullet \to M^\bullet$ is a morphism in $C^-([[RG]])$ which is homotopic to ϕ.*

LEMMA 14.4 (see [21, Lemma VI.8.18]). — *Let $R, R_0 \in \mathrm{Ob}(\mathcal{C})$ be Artinian so that R_0 is a quotient ring of R. Let G be a **finite** group. As in Remark 14.2, we write $X \to X_0$, $\phi \to \phi_0$ for the functor $R_0 \widehat{\otimes}_R -$. Let $M^\bullet$ (resp. $N^\bullet$) be a bounded above complex of abstractly free finitely generated $[RG]$-modules (resp. $[R_0G]$-modules), and let ψ be a quasi-isomorphism $\psi : M_0^\bullet \to N^\bullet$ in $C^-([R_0G])$ which is surjective on terms. Then there exist a bounded above complex $L^\bullet$ of abstractly free finitely generated $[RG]$-modules, a quasi-isomorphism $\phi : M^\bullet \to L^\bullet$ in $C^-([RG])$, and an isomorphism $\rho : L_0^\bullet \to N^\bullet$ in $C^-([R_0G])$, such that $\rho\phi_0 = \psi$.*

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