Coherent Galois module structure and Riemann Roch Theorems

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/Lille2001/talk1.pdf

Three Themes

- Euler characteristics as congruences and compromises.

  Congruences between which objects, and modulo what? Additional assumptions and structures lead to new Euler characteristics.

- Computing Euler characteristics

  Riemann Roch Theorems;
  Backing up into finer K-groups;
  Pushing ambiguities into high codimension
  Adding additional structures (metrics, pairings, comparison maps)

- Applications

  Galois theory
  Geometric proofs of numerical equalities.

Coherent cohomology groups as Galois modules

$\mathcal{O} =$ a regular Noetherian ring.

$X =$ a projective regular scheme over $\mathcal{O}$

$G =$ a finite (constant) group acting on X over $\mathcal{O}$.

$F =$ a coherent $\mathcal{O}_X$-G-module.

Means: On each G-stable open affine Spec(A) of X, F(Spec(A)) is an A-module with an action of G compatible with the action of G on A.

Example: $F = \mathcal{O}_X$; $X =$ Spec($\mathcal{O}_N$) for some Galois G-extension N of $\mathbf{Q}$ when $\mathcal{O} = \mathbf{Z}$.

Basic Problem:

Describe the (finitely generated) $\mathcal{O}G$-modules $H^i(X, F)$ for $i \in \mathbf{Z}$.

Euler characteristics as congruences between modules

$G_0(\mathcal{O}G)$ = Grothendieck group of all finitely generated $\mathcal{O}G$-modules.

Def: $\chi(F) = \sum_{i=0}^{\dim(X)} (-1)^i [H^i(X,F)]$ in $G_0(\mathcal{O}G)$.

Problem: Compute $\chi(F)$ using a minimal amount of information.

Example: (*Question of Pappas*) Suppose the action of G on X is free, and that X is regular and projective over $\mathcal{O} = \mathbf{Z}$. Is $\chi(F)$ the class of a free module in $G_0(\mathbf{Z}G)$?

Answer is "yes" if X is a variety over a finite field (Nakajima) or if X is flat over $\mathbf{Z}$ of dimension 1 (Taylor - Queyrut) or 2 (Pappas).

Pappas has shown that Vandiver's conjecture implies the answer is "yes" if $G = \mathbf{Z}/p$, p is prime and $p > \dim(X)$.

A bestiary of Euler characteristics

Additional hypotheses and structures lead to new Euler characteristics:

1. The action of G on X is **tame** if the wild inertia group of each $x \in X$ is trivial. The Cech cohomology of F can then be computed by a bounded complex $P^\bullet$ of projective $\mathcal{O}G$-modules (a "Noether Theorem"). This leads to $\chi_P(F) = \sum_i (-1)^i [P^i]$ in $K_0(\mathcal{O}G)$

2. Metrics on X and F lead to Arakelov Euler characteristics (Lecture 2).

3. Pairings on cohomology lead to Hermitian Euler characteristics (Lecture 3).

4. Comparison maps to other cohomology groups lead to invariants in relative K-groups (Burns, Flach, Agboola)

Comments on applications

Theorem(Nakajima) If C is a smooth projective curve over a field k and G is a p-group which is the Galois group of a connected étale cover X of C then $\#G^{ab} \le p^{\mathrm{genus}(C)}$.

Sketch There's an exact sequence

$0 \to H^0(\Omega^1_{X/k}) \to P_1 \to P_0 \to H^1(\Omega^1_{X/k}) = k \to 0$

in which the P_i are projective kG-modules. Hence

$$kG^{ab} \cong H^{-2}_{Tate}(G,k) = H^0_{Tate}(G, H^0(\Omega^1_{X/k})).$$

Since $X \to C$ is étale, $\Omega^1_{X/k} = \mathcal{O}_X \otimes_{\mathcal{O}_C} \Omega^1_{C/k}$. So $H^0_{Tate}(G, H^0(\Omega^1_{X/k}))$ is a quotient of

$$H^0(\Omega^1_{X/k})^G = H^0(\Omega^1_{C/k})$$

which has dimension genus(C).

Comment: This is a non-trivial constraint on which G can occur as Galois groups of étale covers of C. Raynaud and Harbater showed, e.g., that all finite p-groups occur as Galois groups over affine curves over $\overline{\mathbf{F}}_p$.

Embedding problems, quadratic invariants

Embedding problems generalize the problem of constructing covers with a given Galois group.

Example: Let K be a field with absolute Galois group G_K. When can one lift an orthogonal representation $\pi : G_K \to O(n)$ to the double cover S$pin(n)$ of $O(n)$? The obstruction is a Stiefel -Whitney class $w_2(\pi)$ in $H^2(G_K, \{\pm 1\})$.

Suppose π results from the permutation representation of G_K on the $n = [N : K]$ embeddings into K^{sep} of a finite separable extension N of K. The trace $tr_{N/K}$ as K-bilinear form has a Hasse-Witt invariant $hw_2(N, tr)$ in $H^2(G_K, \{\pm 1\})$. One has a discriminant $(d_{N/K}) \in H^1(G_K, \{\pm 1\}) = K^*/(K^*)^2$.

Serre(1984) $hw_2(N, tr) = w_2(\pi) + (2) \cup (d_{N/K})$.

One can thus use tr to decide if one can lift π.

Recent work of (Esnault - Viehweg, Kahn) and (Cassou-Noguès - Erez - Taylor) has extended Serre's formula to finite tame covers of projective schemes $X \to Y$ over $\mathbf{Z}[\frac{1}{2}]$. The invariants they consider are in $H^{\bullet}_{et}(Y, \{\pm 1\})$, e.g. in $H^2_{et}(Y, \{\pm 1\})$. The classical case is $Y = \mathrm{Spec}(K)$.

Saito has proved a different generalization when $X \to Y = \mathrm{Spec}(K)$ is the structure morphism of a variety X of even dimension over K. He considers the second Hasse-Witt invariant of the cup product pairing on the middle dimensional de Rham cohomology of X. This is related to the second Stiefel Whitney class of the orthogonal representations of G_K coming from twists of the action of G_K on the middle-dimensional l-adic étale cohomology of X.

Computing Euler characteristics

Riemann-Roch squares

$C(G,S)$ = category of G-schemes over base S

$c : F \to H$ a natural transformation between ring valued functors on $C(G,S)$.

Example(Lefschetz-R-R): T = subgroup of G. $K_0^G(X)$ = Groth. gp. of G-vector bundles. $c : F(X) = K_0^G(X) \to H(X) = K_0^T(X^T)[1/\#T]$ induced by restriction maps.
$F(\pi), H(\pi)$ = Euler characteristic functions associated to a proper $\pi : X \to Y$.

A Riemann Roch square (à la Fulton-Lang) is a "correction factor" $t(\pi)$ for a given morphism $\pi : X \to Y$ in $C(G,S)$ so that there is a commutative diagram:

$$\begin{array}{ccc} F(X) & \overset{t(\pi)\cdot c(X)}{\longrightarrow} & H(X) \\ {\scriptstyle F(\pi)}\downarrow & & \downarrow{\scriptstyle H(\pi)} \\ F(Y) & \overset{c(Y)}{\longrightarrow} & H(Y) \end{array} \qquad (1)$$

Example: (Chevalley-Weil) Suppose $T = < g >$ for some $g \in G$, X^T is a reduced finite set of closed points, $Y = \text{Spec}(k)$ with k a field. Look for RRoch square

$$\begin{array}{ccc} K_0(G,X) & \stackrel{t\cdot\text{restriction}}{\longrightarrow} & K_0^T(X^T)[1/\#T] \\ Euler\downarrow & & \downarrow Euler \\ K_0(G,Y) = G_0(kG) & \stackrel{\text{restriction}}{\longrightarrow} & G_0(kT) \end{array} \quad (2)$$

One has

$$K_0^T(X^T) = \bigoplus_{x \in X^T} G_0(k(x)T)$$

with $G_0(k(x)T) =$ free abelian group on simple $k(x)T$-modules. Take $\text{char}(k) > 0$ (the other case is similar). There is a Brauer trace function

$$BrT(g) : G_0(k(x)T) \to W(k(x)) = \text{Witt vectors}$$

with

$$BrT(g)(c) = \text{sum of lifts of eigenvalues of } g \text{ on } c$$

Suppose X is a smooth projective curve, and the order of g is prime to char(k). For $x \in X^g$ the action of $T =< g >$ on the tangent space m_x/m_x^2 is given by a character $\chi_x : T \to k(x)^*$.
Riemann-Roch square for curves X:

$$\begin{array}{ccc} K_0(G,X) & \stackrel{\bigoplus_x z_x(g)}{\longrightarrow} & \bigoplus_{x\in X^g} W(k(x)) \\ Euler\downarrow & & \downarrow Trace \\ K_0(G,Y) = G_0(kG) & \stackrel{BTr(g|*)}{\longrightarrow} & W(k) \end{array} \tag{3}$$

where

$$z_x(g) = \frac{1}{1-\chi_x(g)} \cdot BTr(g)|i_x^* c$$

$i_x : x \to X$ is inclusion, $i_x^* c$ is a $k(x)$ vector space with g action when c is a locally free O_X-G-module, and g acts trivially on $k(x)$.

To explain (later): The correction factor

$$\frac{1}{1-\chi_x(g)}$$

and its connection to Stickelberger's Theorem.

Corollary: (Nakajima) If the action of G on all of X is free then $X^g = \emptyset$. So

$$BTr(g|\text{Euler}(c)) = 0$$

for all non-identity elements $g \in G$ of order prime to char(k), and all $c \in K_0(G, X)$, where Euler(c) $\in G_0(kG)$. By Brauer representation theory, this forces

$$\text{Euler}(c) = \text{class of a free } k[G] - \text{module.}$$

Lefschetz-RR Theorems arise from combining

- localization sequences
- vanishing theorems ("Segal Concentration")
- explicit computations of boundary maps.

Localization, Concentration and backing up into refined K-groups

When $S = \text{Spec}(\mathcal{O})$, $K_0^G(S)$ is the Grothendieck group of $\mathcal{O}G$ modules which are locally free $\mathcal{O}$-modules.
Choose $W \subset X$ so G action on $X - W$ is free and $(X - W)$ regular.
$K_0^G(X, W)$ = Grothendieck group of bounded complexes of G-vector bundles on X which are exact off W.

ThomasonThe localization sequence

$$K_1^G(X-W) \to K_0^G(X,W) \to K_0^G(X) \to K_0^G(X-W) \to$$

is a long exact sequence of $K_0^G(S)$-modules via $\otimes_{\mathcal{O}}$.

We can thus localize this sequence at prime ideals ρ of $K_0^G(S)$. Say ρ is not I-adic if is is not pulled back from $K_0(S)$.

Example $\mathcal{O} = k$, $g \neq$ id of order prime to char(k), $\rho = \text{Ker}(BTr(g|*) : G_0(kG) \to W(k))$.

"Segal Concentration" (CEPT) If ρ is not I-adic then $K_i^G(X-W)_\rho = 0$ for all i.

Example $X - W = \mathrm{Spec}(L)$, L a field with faithful G action. Then $K_0^G(X-W) =$ Groth. group of modules for the twisted group algebra $L \circ G$. This is infinite cyclic on $[L]$, with $K_0^G(S)$ action factoring through $K_0(S)$. The proof of the Segal concentration Theorem reduces to this case via localization and induction on $\dim(X)$.

Consequence For ρ not I-adic, the sequence

$$K_1^G(X-W)_\rho \to K_0^G(X,W)_\rho \to K_0^G(X)_\rho \to \cdots$$

reduces to an isomorphism

$$K_0^G(X,W)_\rho \to K_0^G(X)_\rho.$$

Basic Problem Find an explicit $T_\rho \in K_0^G(X,W)_\rho$ with image the multiplicative identity $1_\rho = \mathcal{O}_X$ of $K_0^G(X)_\rho$. This gives Riemann-Roch square

$$\begin{array}{ccccc} K_0^G(X)_\rho & \xrightarrow{T_\rho \cdot *} & K_0^G(X,W)_\rho & \xrightarrow{\text{homology}} & G_0^G(W)_\rho \\ \text{Euler}\downarrow & & \downarrow\text{Euler} & & \downarrow\text{Euler} \\ K_0^G(Y)_\rho & \longrightarrow & K_0^G(Y)_\rho & \xrightarrow{\text{forgetful}} & G_0^G(Y)_\rho \end{array}$$

where $G_0^G(W) =$ Groth. group of all coherent O_W-G-modules.

Application: For c a locally free O_X-G module one has

$$\text{homology}(T_\rho \cdot c) = \text{homology}(T_\rho) \cdot c \qquad (4)$$

If I_W is the ideal sheaf of W, some power I_W^n annihilates each homology sheaf of the complex T_ρ. So one can compute (4) from the restriction of c ot the infinitesimal neighborhood $W(n)$ of W having ideal sheaf I_W^n.

An example where $\text{res}_X^W(c)$ suffices:

Suppose $\mathcal{O} = \mathbf{Z}[\zeta_{\#G}]$, q =maximal ideal of $\mathcal{O}$, $g \in G$ has order prime to $\#(\mathcal{O}/q)$.
$\rho = \{\chi \in K_0^G(S) : \chi(g) \in q\}$
$W = \cup\{X^\sigma : \sigma \in G \text{ is conjugate to } g\} \xrightarrow{i} X$
Suppose W regular with ideal sheaf $I_W \subset O_X$, $\mathcal{N}_W = I_W/I_W^2$ = conormal bundle on W.
Then $K_0^G(W) = G_0^G(W)$. By resolving O_W on X with a Koszul resolution, one finds that

$$i^* \circ i_* : K_0^G(W) \to K_0^G(W)$$

is multiplication by

$$\lambda_{-1}(\mathcal{N}_W) = \sum_i (-1)^i \Lambda^i \mathcal{N}_W$$

The localization sequence shows $i_* : K_0^G(W)_\rho \to K_0^G(X)_\rho$ is an isomorphism. So if $i_*(c) = 1_X = T_\rho$, we have

$$O_W = 1_W = i^*(1_X) = i^* i_*(c) = \lambda_{-1}(\mathcal{N}_W) \cdot c$$

This shows

$$c = \frac{1}{\lambda_{-1}(\mathcal{N}_W)} \quad \text{in} \quad K_0^G(W)_\rho.$$

This leads to the square

$$
\begin{array}{ccccc}
K_0^G(X)_\rho & \xrightarrow{\frac{\text{restrict to} W}{\lambda_{-1}(\mathcal{N}_W))}} & K_0^G(W)_\rho & & \\
\text{Euler}\downarrow & & \text{Euler}\downarrow & \overset{\text{Euler}}{\searrow} & \\
K_0^G(Y)_\rho & \xrightarrow{\text{forgetful}} & G_0^G(Y)_\rho & \underset{i'_*}{\longleftarrow} & G_0^G(Y')_\rho
\end{array} \tag{5}
$$

where $Y' = \text{Image}(W \to Y) \xrightarrow{i'} Y$.

Suppose $\dim(Y') = 0$ and $Y'^G = Y'$.
Only classes of free $\mathcal{O}_{Y'}G$-modules have image 0 in $G_0(Y')_\rho$ for all non-I-adic ρ. Thus for c in $K_0^G(X)$ one can find $\text{Euler}(c) \in G_0^G(Y)_\rho$ up to the image of $\text{Ind}_{\{e\}}^G K_0(Y')_\rho$ in $K_0^G(Y)_\rho$ by determining

$$\frac{\text{restriction of } c \text{ to } W}{\lambda_{-1}(\mathcal{N}_W))}$$

in $K_0^G(W)_\rho$ for non-I-adic ρ.

Illustration: Suppose $\#G = l =$ prime, G acts tamely on $X = \text{Spec}(O_N)$ for some number field O_N, and $\zeta_l \in \mathcal{O}$. Then

$$\lambda_{-1}(\mathcal{N}_W)^{-1} = \sum_{w \in W} \frac{1}{1 - \chi w}$$

where W is the finite set of ramified primes of O_N, and χ_w is the character of the finite $\mathcal{O}G$-module I_w/I_w^2. In $K_0^G(S)_\rho$ one has

$$\frac{1}{1 - \chi w} = \frac{-1}{l} \sum_{i=0}^{l-1} i \cdot \chi_w^i = -\theta(\chi_w).$$

$$\text{where} \quad \theta = \frac{1}{l} \sum_{i=0} i\sigma_i$$

is the first Stickelberger element of $\mathbf{Z}[\frac{1}{l}]\text{Aut}(G)$. This and the Lefschetz RRoch square imply: **Thm**(McCulloh): The class $[O_N] = \text{Euler}(\mathcal{O}_X)$ is in the image of θ acting on $K_0^G(S)[1/l]$ up to classes induced from the trivial subgroup.

Higher dimensional X Computing $\lambda_{-1}(\mathcal{N}_W)^{-1}$ leads to higher Stickleberger elements. Taylor's talk will discuss $\dim(X) = 2$.

Gamma filtrations, Chern classes, high codimension strategies

$K_0^G(X)$ has λ and γ operators for $i \geq 0$ defined using power series in t. If V is a vector bundle

$$\lambda_t(V) = \sum_{i=0}^{\infty} \lambda^i(V)t^i = \sum_{i=0}^{\infty} \Lambda^i_{\mathcal{O}_X} V t^i.$$

So $\lambda_t(L) = 1 + Lt$ for L a line bundle. Define

$$\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$$

$$\gamma_t(x) = \sum_{i=0}^{\infty} \gamma^i(x)t^i = \lambda_{t/(1-t)}(x)$$

for all x and y.

The m^{th} term $F^i_{G,X} = F^i_\gamma K_0^G(X)$ in the γ-filtration on $K_0^G(X)$ is additively generated by all products

$$\gamma^{r_1}(x_1)\cdots\gamma^{r_n}(x_n)$$

with $r_i \geq 0$, $\sum_i r_i \geq m$ and

$$x_i \in F^1_{G,X} = \mathrm{Ker}(K_0^G(X) \stackrel{\mathrm{rank}_{\mathcal{O}_X}}{\longrightarrow} \mathbf{Z}).$$

The i^{th} Chern class of a vector bundle V of rank n is

$$c^i(V) = \gamma^i(V - n \cdot \mathcal{O}_X) = (-1)^d \sum_{j=0}^{d} (-1)^j \Lambda^j_{\mathcal{O}_X} V$$

in the i term $F^i_{G,X}/F^{i+1}_{G,X}$ of the Graded K-ring

$$GrK_0^G(X) = \bigoplus_{i=0}^{\infty} F^i_{G,X}/F^{i+1}_{G,X}.$$

Pappas method: Use the following facts.

- Because $X - W \to (X - W)/G = Z$ is étale, pulling back sheaves gives isomorphisms $F^i_{G,X} = F^i_{\{e\},Z} \subset K_0(Z)$. (Morita)

- $F^i_{\{e\},Z} \subset F^i_{\{e\},Z}$(top) = submodule of $K_0^G(Z) = G_0^G(Z)$ generated by coherent sheaves with support of codimension $\geq i$. $F^{d+2}_{\{e\},Z} = 0$ if $\dim(Z) = d + 1$.

Example: Suppose $\dim(X) = d + 1$, G acts tamely on X, $\pi : X \to X/G$ is the projection, $Y = Y^G$ and $V = \pi^* E$ for some vector bundle E of rank d on X/G. Then

$$c^d(V) = \sum_{j=0}^{d} (-1)^j \Lambda^j_{\mathcal{O}_X} V = \pi^*(c^d(E))$$

where

$$c^d(E) = \sum_j m_j F_j$$

for some $m_j \in \mathbf{Z}$ and sheaves F_j supported on a codimension d subschemes $Z_j \subset X/G$, Then

$$\text{Euler}(c^d(V)) = \sum_j m_j \cdot \text{Euler}(\pi^* F_j) \quad \text{in} \quad K_0^G(Y)$$

where $\pi^* F_j$ is a G sheaf on the dimension $d+1-d = 1$ scheme $\pi^* Z_j$. Classical Fröhlich theory works well on dimension 1 schemes. This is the strategy to be discussed in the next lecture in order to compute

$$\text{Euler}(c^d(\Omega^1_{\log}(X/\mathbf{Z})))$$

using ϵ-factors in functional equations.

Example: Suppose $\mathcal{O} = \mathbf{Z}$
$\#G = l$ is prime, and G acts tamely on X,
$X/\mathbf{Z}$ is flat of relative dimension d.
Each $c \in K_0^G(X)$ then has a projective Euler characteristic

$$\chi_P(c) \in Cl(\mathbf{Z}G) = K_0(\mathbf{Z}G)/\text{frees} \cong Cl(\mathbf{Z}[\zeta_l])$$

Thm(Pappas, Taylor, C) For $\#G = l$, there is a proper closed subset $\tilde{W}$ of X independent of c such that the restriction of c to $\tilde{W}$ determines $\chi_P(c)$ modulo the direct sum A_d over $3 \leq i \leq d$ of the i^{th} Teichmuller eigenspace of the l-torsion $Cl(\mathbf{Z}[\zeta_l])[l^\infty]$. (Note $A_d = \{0\}$ if $d = 2$.)

If one considers more general G, it's not known how to find $\chi_P(c)$ exactly from fibral information even when $d = 2$. For G abelian there is more one can prove using "cubic structures":
Taylor's lecture For all abelian G, $d = 2$ and $c = \mathcal{O}_X$, one has an exact expression for $\chi_P(\mathcal{O}_X)$ in $Cl(\mathbf{Z}G)$ using only intersection numbers of components of singular fibers.

Additional techniques used in the example: Compute $L \otimes_{\mathbf{Z}} c \in K_0^G(R \otimes X)$ when

$$L \in F^{d+1}_{G,\mathrm{Spec}(R)}$$

and R an auxiliary Dedekind ring. This forces

$$L \otimes_{\mathbf{Z}} c \in F^{d+1}_{G,R\otimes X}$$

Using Thomason's localization sequence, and one shows one can back this class up into the relative group

$$K_0^G(R \otimes X, R \otimes (W \cup W')) \tag{6}$$

for some finite set of closed points $W' \subset W$. Since $W \cup W'$ is now fibral, we can compute Euler characteristics of classes in (6) using non-I-adic primes, with ambiguities a sum of class of the form $R/m[G]$ with m a prime of R. One then must analyze how much information is obtained from the resulting values for

$$\chi_P(L \otimes_{\mathbf{Z}} c) = L \otimes_R \chi_P(c) \in Cl(RG).$$