

Arakelov Euler Characteristics

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Overview

Talk 1: This talk focused on coherent Euler characteristics of G -sheaves F on projective schemes X having an action of a finite group G . We used Riemann-Roch Theorems and Chern classes to study the Euler map

$$K_0^G(X) \xrightarrow{\pi_*} K_0^G(Y)$$

when $\pi : X \rightarrow Y$ is a proper morphism of Noetherian G -schemes and G is a finite group.

This Talk: We'll add additional structure by metrizing X and F . This leads to Arakelov Euler characteristics, and the use of arithmetic Riemann-Roch results.

Talk 3: We'll add the additional structure of pairings on cohomology provided by Serre-Grothendieck duality Theorems. This leads to hermitian Euler characteristics in Fröhlich's hermitian classgroup.

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The classical picture: rings of integers

Suppose N/K is a finite Galois tame extension of number fields.

Taylor (Fröhlich's Conjecture) The class $[O_N]$ of O_N in $K_0(\mathbf{Z}G)$ can be determined from $[N : \mathbf{Q}]$ and the root numbers $W(V) = \pm 1$ of the irreducible symplectic representations V of G .

But $[O_N]$ doesn't determine the $W(V)$.

Fröhlich suggested considering invariants of O_N together with the trace pairing $(x, y) \rightarrow \text{Tr}_{N/\mathbf{Q}}(xy)$. This leads to local and global Hermitian discriminants of (O_N, Tr) in "Hermitian classgroups." Cassou-Noguès and Taylor showed these discriminants were enough to determine both local and global symplectic root numbers. This approach will be generalized to projective schemes over \mathbf{Z} in lecture 3.

One application: Unconditional geometric proofs that various root numbers are positive, as predicted by other conjectures (Birch-Sw-Dyer).

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The Arakelov approach for rings of integers Instead of the trace pairing, consider the G -equivariant Hecke pairing

$$h^{\text{Hecke}} : (\mathbf{C} \otimes N) \times (\mathbf{C} \otimes N) \rightarrow \mathbf{C}$$

defined by

$$h^{\text{Hecke}}(\lambda \otimes m, \nu \otimes n) = \lambda \bar{\nu} \sum_{\sigma} \sigma(m) \overline{\sigma(n)}$$

where the sum is over the embeddings $\sigma : N \rightarrow \mathbf{C}$. This form arises from the volume form defined by the Euclidean absolute values of N . It was used by Hecke in his proof of the functional equation for L-functions.

In the Arakelov approach, one considers the class of $[O_N, h^{\text{Hecke}}]$ in an appropriate G -equivariant Arakelov class group $A(\mathbf{Z}G)$. This class is really a metrized G -equivariant Euler characteristic. We'll show it determines (and is determined by) the symplectic root numbers $W(V)$.

Goal today: Generalize $[O_N, h^{\text{Hecke}}]$ and its connection to symplectic root numbers to projective schemes X having a tame action of G .

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Some related topics:

Classically there are also results on trace forms restricted to the square root $\mathcal{D}_{N/\mathbb{Q}}^{-1/2}$ of the inverse different of N . On schemes X there is a literature concerning analogous “quadratic bundles,” which have hermitian and Arakelov characteristics.

Various height functions on subschemes of a fixed projective scheme over \mathbb{Z} can be rephrased in terms of metrized Arakelov Euler characteristics. Often these subschemes have natural actions of finite groups or group schemes (e.g. when they are Grassmanians). In these cases it’s useful to have metrized G -equivariant Euler characteristics.

Definition of Arakelov Euler characteristics

Standard Hypotheses

- $X =$ a regular scheme, flat and projective over \mathbb{Z} .
- $G =$ a finite group acting tamely on X . Recall that this means the order of the inertia group $I_x \subset G$ of each point $x \in X$ is prime to the residue characteristic of x .
- $Y = X/G$ is also regular, and the irreducible components of finite fibers of Y are smooth with normal crossings and multiplicities prime to the residue characteristic.

Let h be a G -equivariant Kähler metric on $X(\mathbb{C})$ invariant under complex conjugation. So h is defined by a closed $(1, 1)$ -form; one needs this for Arakelov theory to work.

Definition 1 A G -hermitian bundle on X is a pair (E, k) for which:

E is a locally free O_X -module having an action of G compatible with the G -action on X ;

k is a G -invariant smooth hermitian metric on the bundle $E_{\mathbb{C}}$ induced by E on $X(\mathbb{C})$, and k is invariant under complex conjugation.

Example: Assume X satisfies the Standard Hypotheses. Let E be the sheaf $\Omega_X^1(\log)$ of log-differentials on X with respect to the reduction of the union of the bad fibers of X . This agrees with Ω_X^1 on the general fiber, so we can let k result from the Kähler metric h .

$\phi \in \hat{G} =$ irreducible complex characters of G

$V_{\phi} =$ the ϕ -isotypic part of a $\mathbb{C}G$ -module V

Let h be a G -equivariant Kähler metric $X(\mathbb{C})$ invariant under complex conjugation.

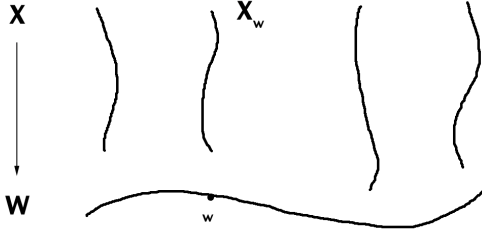
Definition 2 The ϕ -determinant of cohomology of a locally free O_X -module E is the complex line

$$\det(H^*(E)_{\phi}) = \otimes_i \det(H^i(X(\mathbb{C}), E_{\mathbb{C}})_{\phi})^{(-1)^i}$$

The Quillen-Bismut metric on this line induced by a metric k on E will be written k_{ϕ} .

Rationale for Quillen metrics

For simplicity, take G trivial. Suppose $X(\mathbb{C})$ is the fiber \mathcal{X}_w at $w \in \mathcal{W}$ of a smooth projective family $\mathcal{X} \rightarrow \mathcal{W}$ of complex varieties, and that $E = \mathcal{E}_w$ for some vector bundle \mathcal{E} on \mathcal{X} . Suppose \mathcal{E} has a smooth metric extending that of E and that there is a smooth Kähler metric on \mathcal{X} extending that of X .



One can construct algebraically a line bundle $\det(H^*(\mathcal{E}))$ on \mathcal{W} which specializes to $\det(H^*(E))$ at w . The Quillen metric on $\det(H^*(E)) = \det(H^*(\mathcal{E}))_w$ extends to a metric on $\det(H^*(\mathcal{E}))$ which varies **smoothly** with w .

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Quillen metrics and analytic torsion

The first approximation to constructing such a metric is to use the L^2 -metrics on the harmonic forms

$$\mathcal{H}^q(\mathcal{X}_w, \mathcal{E}_w) = \ker(\Delta_w^q)$$

where Δ_w^q is the Laplace operator on $(0, q)$ forms $A_w^q = A^{0,q}(\mathcal{X}_w, \mathcal{E}_w)$. This won't in general vary smoothly with w because the individual $\mathcal{H}_w^q = \mathcal{H}^q(\mathcal{X}_w, \mathcal{E}_w)$ don't vary smoothly.

Example:(Faltings) Let $C =$ complex curve. For $m \gg 0$, let $\mathcal{X} \rightarrow \mathcal{W} = \text{Pic}^m(C)$ be the constant family with fiber C . Define \mathcal{E} by letting \mathcal{E}_w be the line bundle on \mathcal{X}_w associated to $w \in \text{Pic}^m(C)$. The dimensions of $\mathcal{H}^0(\mathcal{X}_w, \mathcal{E}_w)$ and of $\mathcal{H}^1(\mathcal{X}_w, \mathcal{E}_w)$ can jump for special w . But by Riemann-Roch the difference of these dimensions is the constant $1 - \text{genus}(C) + m$.

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Let $A_{w,\lambda}^q \subset A_w^q$ be the eigenvectors for Δ_w^q of eigenvalue λ . Suppose $\dim(\mathcal{X}_w) = 1$. For $\lambda > 0$, $\bar{\delta} : A_{w,\lambda}^0 \rightarrow A_{w,\lambda}^1$ is an isomorphism which multiplies L^2 norms by $\sqrt{\lambda}$, since

$$\langle \bar{\delta}v, \bar{\delta}v \rangle_{L^2} = \langle \bar{\delta}^* \delta v, v \rangle_{L^2} = \langle \Delta_w^q v, v \rangle_{L^2} = \lambda \langle v, v \rangle_{L^2}$$

Thus

$$\bar{\delta} : \text{Det}(A_{w,\lambda}^0) \rightarrow \text{Det}(A_{w,\lambda}^1)$$

is an isomorphism of one dimensional \mathbb{C} -vector spaces which multiplies L^2 norms by $\lambda^{\dim(A_{w,\lambda}^1)/2}$. Because of this, the natural inclusions

$$\mathcal{H}_w^q = A_{w,0}^q \rightarrow A_{w,<b}^q = \bigoplus_{\lambda < b} A_{w,\lambda}^q$$

for $b > 0$ and $q = 0, 1$ induce an isomorphism

$$\frac{\text{Det}(\mathcal{H}_w^0)}{\text{Det}(\mathcal{H}_w^1)} \xrightarrow{\tau} \frac{\text{Det}(A_{w,<b}^0)}{\text{Det}(A_{w,<b}^1)}$$

This τ is an isometry for the L^2 norms after we multiply on the L^2 norm on the right (which varies smoothly near w) by

$$\prod_{0 < \lambda < b} \left(\frac{1}{\sqrt{\lambda}} \right)^{\dim(A_{w,\lambda}^1)}$$

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Conclusion: Let

$$T_{w,\mu} = \prod_{\mu < \lambda'} \left(\frac{1}{\sqrt{\lambda'}} \right)^{\dim(A_{w,\lambda'}^1)}$$

Multiplying the L^2 -norms on the domain and range of τ by $T_{w,0}$ and $T_{w,b}$, respectively, makes τ an isometry. Letting $b \rightarrow \infty$, we get that the norm

$$(L^2 \text{ Norm}) \times T_{w,0} \quad \text{on} \quad \frac{D\mathcal{H}_w^0}{D\mathcal{H}_w^1} = \frac{DA_{w,0}^0}{DA_{w,0}^1}$$

varies smoothly with w . This is the Quillen norm. The analytic torsion $T_{w,0}$ is defined rigorously via this zeta function of a complex variable s :

$$\zeta_{\Delta_w^1}(s) = \sum_{0 < \lambda} \lambda^{-s \dim(A_{w,\lambda}^1)}$$

The function

$$\prod_{0 < \lambda} \lambda^{\dim(A_{w,\lambda}^1)} = T_{w,0}^{-2} = \exp(-\zeta_{\Delta_w^1}(0))$$

is the regularized determinant of the Laplacian Δ_w^1 .

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Multiplicative Arakelov degrees

$(L, k_0) =$ Hermitian line bundle on $\text{Spec}(\mathbf{Z})$.

Choose l_0, l_p for all primes p so $\mathbf{Q} \otimes L = \mathbf{Q}l_0$,
 $\mathbf{Z}_p \otimes L = \mathbf{Z}_p l_p$.

Then $l_0 = \alpha_p l_p$ for some $\alpha_p \in \mathbf{Q}_p^*$.

Definition 3

$$\deg(L, k_0) = \left(\prod_p \alpha_p \right) \times k_0(l_0)$$

in the Arakelov divisor group

$$A(\mathbf{Z}) \stackrel{\text{def}}{=} \frac{J(\mathbf{Q})}{\mathbf{Q}^* \cdot \prod_{p < \infty} \mathbf{Z}_p^*} = \mathbf{R}_{>0}$$

Example: Picking $l \in L$ so $L = \mathbf{Z} \cdot l$, we can take $l_0 = l_p = l$ for all p . Then $\deg(L, k_0) = k_0(l)$ is the length of the generator l . Note that the logarithm gives an isomorphism from $A(\mathbf{Z})$ the first arithmetic Chow group of Gillet-Soulé $\widehat{CH}^1(\text{Spec}(\mathbf{Z})) = \mathbf{R}$.

Equivariant generalization

$P^\bullet =$ perfect complex of $\mathbf{Z}G$ -modules

$j_\bullet = \{j_\phi\}_{\phi \in \widehat{G}}$ metrics on $\det((H^\bullet(P^\bullet) \otimes \mathbf{C})_\phi)$.

Choosing local and global bases gives $\deg(P^\bullet, j_\bullet)(\phi) \in J(\overline{\mathbf{Q}})$ for $\phi \in \widehat{G}$ and a class

$$\deg(P^\bullet, j_\bullet) \in \frac{\text{Hom}_\Gamma(R_G, J(\overline{\mathbf{Q}}))}{\text{denominator}} \stackrel{\text{def}}{=} A(\mathbf{Z}G)$$

where $\Gamma = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

Definition 4 The Arakelov Euler characteristic of a hermitian G -bundle (E, k) is

$$\chi_A(E, k) = \deg(R\Gamma(X, E), k_{Q, \bullet}) \in A(\mathbf{Z}G)$$

Here the total cohomology $R\Gamma(X, E)$ is represented by a perfect complex of $\mathbf{Z}G$ -modules because the G action on X is tame. $k_{Q, \bullet}$ is the Quillen metric on the determinant of the isotypic pieces of $R\Gamma(X, E)$.

Note: This extends from (E, k) to complexes of hermitian G -bundles.

Arakelov Euler characteristics and Epsilon Factors

$R_G^s =$ subgroup of virtual symplectic characters in R_G .

Restricting to R_G^s gives a quotient $A^s(\mathbf{Z}G)$ of $A(\mathbf{Z}G)$.

Proposition 5 (Cassou-Noguès and Taylor) $A^s(\mathbf{Z}G)$ contains a subgroup $\text{Hom}_\Gamma(R_G^s, \mathbf{Q}^*)$ of "rational classes".

For $\phi \in R_G$, Deligne defines a constant $\epsilon_0(X/G, \phi)$ using the local ϵ constants associated to the Euler characteristic of $H_{\text{étale}}^\bullet(X \times \overline{\mathbf{Q}}_p)$ and of $H_{\text{Hodge}}^\bullet(X)$.

For simplicity we add the following to the Standard Hypotheses:

(*) The fibers of Y over \mathbf{Z} are reduced.

Let h stand (also) for the metrics on the terms $\Lambda^i \Omega_X(\log)$ of the log de Rham complex $\Lambda^\bullet \Omega_X(\log)$ which result from a suitably normalized Kähler metric h on X .

Let

$$\chi_A^s(\Omega_X^\bullet(\log), h_{Q, \bullet}) = \text{image of } \chi_A(\Omega_X^\bullet(\log), h_{Q, \bullet}) \text{ in } A^s(\mathbf{Z}G)$$

Theorem 6 (CPT) Assuming (*), one has

$$\chi_A^s(\Omega_X^\bullet(\log), h_{Q, \bullet}) = \epsilon_0^s$$

where $\epsilon_0^s \in \text{Hom}_\Gamma(R_G^s, \mathbf{Q}^*)$ is the rational class in $A^s(\mathbf{Z}G)$ defined by the character function

$$\phi \rightarrow \epsilon_0(X/G, \phi)$$

for $\phi \in R_G^s$.

Outline of the proof

Let $E = \Omega_X^1(\log)$. Suppose G is trivial. Then $X = Y$ and $\Lambda^d E = \omega_{X/Z} =$ dualizing sheaf because of (*). The compatibility of Serre duality with L^2 norms on harmonic forms implies

$$\deg(T_{d-i}, h_{L^2, \bullet}) = \deg(T_i, h_{L^2, \bullet})^{(-1)^{d+1}}$$

for all i in $A(\mathbf{Z}) = \mathbf{R}$, where $T_i = R\Gamma(X, \Lambda^i E)$. This leads to

$$\chi_A^s(\Omega_X^\bullet(\log), h_{L^2, \bullet}) = \prod_i \deg(T_i, h_{L^2, \bullet})^{(-1)^i} = 1$$

Soulé showed us how to use the Arithmetic Riemann Roch Theorem to prove

$$\chi_A^s(\Omega_X^\bullet(\log), h_{L^2, \bullet}) = \chi_A^s(\Omega_X^\bullet(\log), h_{Q, \bullet})$$

One can also show this with results of Ray-Singer on analytic torsion. Finally, Saito proved

$$\epsilon_0^s = 1$$

under hypothesis (*). Thus

$$\chi_A^s(\Omega_X^\bullet(\log), h_{Q, \bullet}) = 1 = \epsilon_0^s$$

which proves the Theorem when G is trivial.

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For general G , one uses the result for trivial G to reduce the problem to considering characters of degree 0. Furthermore, $E = \pi^* E_Y$ when $E_Y = \Omega_Y^1(\log)$ and $\pi : X \rightarrow Y$ is the projection. So $c^d(E) = \pi^*(c^d(E_Y))$ where the d^{th} Chern class

$$c^d(E_Y) = (-1)^d \sum_{j=0}^d (-1)^j \Lambda^j E_Y$$

lies in the d term F_Y^d of the γ -filtration on $K_0(Y)$. Since F_Y^d is contained in the d -th term of the topological filtration on $K_0(Y)$, we can write

$$c^d(E) = \pi^* c^d(E_Y) = \sum_j m_j \cdot \pi^* F_j \quad (1)$$

in $K_0^G(X)$ where F_j is a sheaf supported on a dimension $d + 1 - d = 1$ scheme $Z_j \subset Y$. To compare the metrized Euler characteristics of both sides of (1), one needs an important Lefschetz Riemann Roch result of Bismut. This reduces the problem to one on metrized one-dimensional schemes, where one can apply resolvent theory à la Fröhlich and Taylor.

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Comments

- 1 There is a correction factor at the trivial character if one assumes only that the fibral multiplicities of Y are prime to the residue characteristic.
- 2 When G is trivial this proves a conjecture of Bloch under our hypotheses. An independent proof of Bloch's conjecture under the same hypotheses was given by Arai. Kato and Saito have a stronger result showing Bloch's conjecture without assumptions about the fibral multiplicities of Y .
- 3 The Theorem shows that the metrized de Rham Euler characteristic $\chi_A^s(\Omega_X^\bullet(\log), h_{Q, \bullet})$ determines the constants $\epsilon_0(X/G, \phi)$ for symplectic ϕ . This is because $\text{Hom}_\Gamma(R_G^s, \mathbf{Q}^*)$ injects into $A^s(\mathbf{Z}G)$.

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- 4 Resolvent theory on 2-dimensional schemes will be discussed by Taylor. If one has a good method for computing (metrized) Euler characteristics on dimension j subschemes of an arbitrary scheme X , it can be used to study (metrized) classes of complexes $\pi^* c$ when c lies in $F^{d-j} K_0(Y)$.
- 5 One can prove an Arakelov Euler characteristic result for Ω_X^\bullet using Dold-Puppe exterior powers of a bounded perfect complex L^\bullet of O_X - G -modules quasi-isomorphic to Ω_X^1 . This result involves ϵ -factors over X/G and the branch locus of $X \rightarrow X/G$ as well as ϵ_0 factors over X/G .

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6 D. Burns has suggested a common refinement of the Arakelov and hermitian Euler characteristics of $\Omega_X^\bullet(\log)$ using the comparison isomorphisms between de Rham and Betti cohomology.

7 There are other natural Arakelov (and hermitian) Euler characteristics to consider. Suppose, for example, that one has a G -equivariant embedding $\iota: X \rightarrow P_{\mathbf{Z}}^N$ and one wants to define the G -height of X with respect to ι . (We will assume G acts on $\mathcal{O}_{P^N}(1)$.)

Faltings heights for trivial G

Give $P_{\mathbf{Z}}^N$ the Fubini-Study metric. Choose a Green's current g_X so $dd^c g_X + \delta_X$ is harmonic and g_X has trivial harmonic projection. Let

$$\text{height}(X) = f_*(\hat{X} \cdot \hat{c}_1(\overline{\mathcal{O}(1)})^{d+1}) \in \mathbf{R}$$

where:

$\hat{X} \in \hat{C}H^{N-d}(P^N)$ is the class of the cycle (X, g_X) ; $\hat{c}_1(\overline{\mathcal{O}(1)})^{d+1} \in \hat{C}H^1(P^N)$ is the first arithmetic Chern class of $\overline{\mathcal{O}(1)}$ with Fubini metric on P^N ;

$$f_*: \hat{C}H^N(P^N) \otimes \mathbf{Q} \rightarrow \hat{C}H^1(\text{Spec}(\mathbf{Z})) = \mathbf{R}$$

is the direct image.

Comments :

- The definition generalizes the classical degree of the generic fiber $X_{\mathbf{Q}}$ in $P_{\mathbf{Q}}^N$. It agrees with the usual log-height of a rational point in projective space.

- One can replace $P_{\mathbf{Z}}^1$ by other spaces on which one has canonical metrized line bundles or vector bundles, e.g. Grassmannians and flag varieties. Many authors have worked on the "arithmetic Schubert calculus" needed to compute intersections in the arithmetic chow group of such spaces. (Gillet-Soulé; Philippon, Cassaigne and Mailhot, Tamvakis, Kaiser-Köhler-Roessler,...) This often leads to interesting rational numbers, e.g. for the heights of flag varieties under various Plücker embeddings.

- Köhler and Roessler have proved an Arakelov theoretic Lefschetz - Riemann Roch result when one has a finite diagonalizable group acting on a scheme X . They and Kaiser have shown how this can be useful when studying non-equivariant heights of flag varieties, using the actions of subgroups of maximal tori. Diagonalizable group actions make it possible to take eigenspaces "over \mathbf{Z} ."

Arakelov heights for non-trivial constant groups G :

Here is one example of such a height when G acts tamely on $X \rightarrow P_{\mathbf{Z}}^1$ and on $\mathcal{O}_{P^N}^1$. The Grassmannian $G_{\mathbf{Z}}^{m,n}$, which represents the functor Schemes \rightarrow Sets defined by

$$T \rightarrow \begin{array}{l} \text{isomorphism classes of locally} \\ \text{free rank } n \text{ quotients of } \mathcal{O}_T^{n+m} \end{array}$$

Both $G_{\mathbf{Z}}^{dn,d}$ and the tautological quotient bundle $\overline{Q}_{dn,d}$ of rank d on $G_{\mathbf{Z}}^{dn,d}$ have canonical metrics. Now use the embeddings

$$X \xrightarrow{\text{diagonal}} (P_{\mathbf{Z}}^n)^d = (G_{\mathbf{Z}}^{n,1})^d \xrightarrow{\text{canonical}} G_{\mathbf{Z}}^{dn,d}$$

Pulling back $\overline{Q}_{dn,d}$ to X gives a line bundle E on X with hermitian metric h . Using the induced Kähler metric on X we set

$$\text{height}(X, G) = \chi_A(E, h_{\text{Quillen}}) \quad \text{in } A(\mathbf{Z}G)$$

One Goal: Use heights associated to constant group actions to derive additional information about non-equivariant heights.