Hermitian Euler Characteristics

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July 2001

http://www.math.upenn.edu/~ted/talks

/Lille2001/talk3.pdf

Overview

Talk 1: This talk focused on coherent Euler characteristics of G-sheaves F on projective schemes X having an action of a finite group G.

Talk 2: We added additional structure by metrizing X and F. This leads to Arakelov Euler characteristics, which have applications to computing ϵ-constants and heights.

Talk 3: We'll add the additional structure of pairings on cohomology provided by Serre-Grothendieck duality Theorems. This leads to hermitian Euler characteristics in Fröhlich's hermitian classgroup. The main application today will be to prove in some cases a prediction of the Birch Swinnerton Dyer concerning signs of symplectic epsilon factors.

Hermitian discriminants

A Hermitian G-module is a pair (M, h) consisting of a finitely generated locally free $\mathbf{Z}G$-module M and a symmetric non-degenerate G-equivariant pairing $h : M_{\mathbf{Q}} \times M_{\mathbf{Q}} \to \mathbf{Q}$, where $M_{\mathbf{Q}} = \mathbf{Q} \otimes M$.

Example: Let $M = O_N$ for N/K a finite tame Galois G-extension of number fields, and let

$$h(x, y) = Tr_{N/\mathbf{Q}}(xy) : N \times N \to \mathbf{Q}.$$

We'll recall the definition of Fröhlich's hermitian discriminant of (O_N, h), which Cassou-Noguès and Taylor proved determines the root numbers of symplectic representations of G.

Let $\Gamma = Gal(\overline{\mathbf{Q}}/\mathbf{Q})$, and R_G^s is the subgroup of symplectic characters in R_G. The group $\mathrm{Det}^s(U(\mathbf{Z}G))$ consist of those functions in

$$\mathrm{Hom}_\Gamma(R_G^s, J(\overline{\mathbf{Q}}))$$

which are determinants of unit ideles of $\mathbf{Z}G$.

Theorem 1 (Fröhlich) *To* (M, h) *one can associate a discriminant* $d(M, h)$ *in the Adelic Hermitian classgroup*

$$AdHCL(\mathbf{Z}G) = \frac{\mathrm{Hom}_\Gamma(R_G^s, J(\overline{\mathbf{Q}}))}{\mathrm{Det}^s(U(\mathbf{Z}G))}$$

Example:

If G is trivial, let $d_h = \mathrm{Det}((h(m_i, m_j))) \in \mathbf{Q}^*$ be the usual discriminant of h for (any) basis $\{m_i\}_i$ of M. Then $d(M, h)$ goes to $1 \times d_h$ under the isomorphism

$$AdHCl(\mathbf{Z}) = \prod_p (\mathbf{Z}_p^*/(\mathbf{Z}_p^*)^2) \times \mathbf{Q}^*$$

induced by evaluation functions on twice the trivial character.

Generalization to complexes

$P^\bullet =$ a perfect complex of $\mathbf{Z}G$-modules

$\langle\ ,\ \rangle = \{\langle\ ,\ \rangle_t\}_t =$ a family of G-equivariant perfect pairings $\langle,\ \rangle_t : H^t(P^\bullet_{\mathbf{Q}}) \times H^{-t}(P^\bullet_{\mathbf{Q}}) \to \mathbf{Q}$ on the cohomology of $P^\bullet_{\mathbf{Q}} = \mathbf{Q} \otimes P^\bullet$ such that $\langle x, y\rangle_t = \langle y, x\rangle_{-t}$.

Proposition 2 *There is a natural discriminant $d(P^\bullet, \langle\ ,\ \rangle) \in AdHCL(\mathbf{Z}G)$ generalizing $d(M, h)$.*

Variants: Instead of working with $\mathbf{Z}G$-modules, one can work with $\mathcal{O}G$-modules when $\mathcal{O}$ is a field or the integers of a number field or a p-adic local field. One has

$$d(P^\bullet, \langle\ ,\ \rangle) \in AdHCL(\mathcal{O}G) = \begin{array}{c}\text{Hom}-\\ \text{description}\end{array}$$

for $P^\bullet$ a perfect complex of $\mathcal{O}G$-modules with perfect pairings $\langle\ ,\ \rangle = \{\langle\ ,\ \rangle_t\}_t$ as above from $H^t(P^\bullet_F) \times H^{-t}(P^\bullet_F)$ to $F = \text{frac}(\mathcal{O})$

de Rham discriminants

Suppose now that $\mathcal{O} = \mathbf{Z}$ or $\mathbf{Z}_p$ for some prime p. Assume

- X is a regular, flat and projective scheme over $\mathcal{O}$ having a tame action by a finite group G, with G acting freely on $X_F = F \otimes_{\mathcal{O}} X$.

- $Y = X/G$ is regular, with smooth irreducible fiber components which cross normally and have multiplicities prime to the residue characteristic.

For simplicity we will also assume:

* The fibers of $Y = X/G$ over $\mathbf{Z}$ are reduced.

** $\dim(X) = 2$.

Let $S = \{p\}$ if $\mathcal{O} = \mathbf{Z}_p$. In the global case $\mathcal{O} = \mathbf{Z}$, let S be any finite set of finite places which includes the v for which the fiber of X_v is not smooth. Define $\omega_{X,S}[1]$ on X to be the two-term complex of G-sheaves

$$[O_X \to \omega_X(X_S^{red})][1]$$

Thus O_X occurs in degree -1. In degree 0 one has the twist $\omega_X(X_S^{red})$ of the relative dualizing sheaf of X by the sum of the reduced fibers of X over primes in S.

Serre duality on the general fiber of X induces family $\langle\ ,\ \rangle$ of perfect pairings on a perfect complex $R\Gamma(X, \omega_{X,S}[1])$ of $\mathcal{O}G$-modules representing the hypercohomology of $\omega_{X,S}[1]$.

Let $d_0(X, G) = d(R\Gamma(X, \omega_{X,S}[1]), \langle\ ,\ \rangle)$ be the associated discriminant in $AdHCL(\mathcal{O}G)$.

We want to connect this discriminant to epsilon factors of symplectic representations.

In the local case $\mathcal{O} = \mathbf{Z}_p$ define

$\epsilon_{S,0}(Y,V) = \epsilon_{p,0}(Y,V) =$ Deligne ϵ_0 constant

In the global case $\mathcal{O} = \mathbf{Z}$, let

$$\epsilon_{S,0}(Y,V) = \prod_{v \in S} \epsilon_{v,0}(Y,V) \cdot \prod_{v \notin S} \epsilon_{v,0}(Y, V - \mathrm{dim}(V) \cdot 1)$$

Define

$$\epsilon^s_{S,0}(Y) \in Hom_{\Gamma}(R^s_G, \mathbf{Q}^*)$$

by

$$\phi \to \epsilon_{S,0}(Y,\phi) \quad \text{for} \quad \phi \in R^s_G$$

Point: In the global case, $\prod_{\text{all } v} \epsilon_{v,0}(Y,V)$ does not converge on V which don't have degree 0, so one has choose an S. Deligne's $\epsilon_{v,0}$ constants are defined via the Galois representations provided by the V parts of $H^\bullet_{et}(\overline{\mathbf{Q}} \otimes X, \mathbf{Q}_l)$.

There is an embedding

$$\Delta_f : \mathbf{Q}^* \to J(\mathbf{Q})$$

which is the diagonal embedding on positive rationals, and which sends -1 to the idele $(-1)_f$ having component -1 at all finite places and 1 at the archimedean place. We will identify

$$\Delta_f(\epsilon_0^s) \in \mathrm{Hom}_\Gamma(R_G^s, J(\mathbf{Q}))$$

with the class it defines in

$$AdHCL(\mathbf{Z}G) = \frac{\mathrm{Hom}_\Gamma(R_G^s, J(\overline{\mathbf{Q}}))}{\mathrm{Det}^s(U(\mathbf{Z}G))}$$

Let $B' \in \mathrm{Hom}(R_G^s, J(\mathbf{Q}))$ be defined by

$$B'(\chi_V) = (-1)_{\mathcal{O}}^{\mathrm{dim}(V)/2}$$

where $(-1)_{\mathcal{O}}$ is -1 embedded diagonally (resp. at p) into $J(\mathbf{Q})$ when $\mathcal{O} = \mathbf{Z}$ (resp. $\mathcal{O} = \mathbf{Z}$). Define B to be the image of B' in $Ad\ HCl(\mathbf{Z}G)$.

Finally we have to push the de Rham discriminant

$$d_0(X, G) \in AdHCL(\mathcal{O}G)$$

into $AdHCL(\mathbf{Z}G)$ using a map

$$t_{\mathcal{O}} : AdHCl(\mathcal{O}G) \to AdHCL(\mathcal{Z}G).$$

If $\mathcal{O} = \mathbf{Z}$ take $t_{\mathbf{Z}}$ to be the identity map. If $\mathcal{O} = \mathbf{Z}_p$, we define $t_{\mathbf{Z}_p}$ to be the map on on Hom-descriptions induced by the natural embedding $\overline{\mathbf{Q}} \otimes \mathbf{Q}_p \to J(\overline{\mathbf{Q}})$.

Theorem 3 (CPT) *For X as above (of dimension 2),*

$$t_{\mathcal{O}}(d_0(X, G)) = B^{\chi(Y)} \Delta_f(\epsilon^s_{S,0}(Y)) \quad \text{in} \quad Ad\,HCl(\mathbf{Z}G)$$

where $\chi(Y) = \mathrm{rank}_{\mathcal{O}} H^0(O_Y) - \mathrm{rank}_{\mathcal{O}} H^1(O_Y)$.

Corollary: $t_{\mathcal{O}}(d_0(X, G))$ and $\chi(Y)$ determine all the symplectic $\epsilon_{S,0}$-constants of X.

Proof of Corollary: A result of Cassou-Noguès and Taylor shows $\Delta_f(\epsilon^s_{S,0}(Y))$ determines $\epsilon^s_{S,0}(Y)$.

Comments

1 There is an analogous result concerning the shifted de Rham complex $\Omega^{\bullet}_X[1] = [O_X \to \Omega^1_X][1]$. This result involves ϵ-factors rather than ϵ_0 factors.

2 For X of arbitrary dimension $d+1$, one can construct a de Rham discriminant $d_{dR}(X, G) \in AdHCl(\mathbf{Z}G)$ using Dold-Puppe exterior powers of a locally free resolution of Ω^1_X. The image of $d_{dR}(X)$ in the ordinary classgroup $Cl(\mathbf{Z}G)$ is the equivariant Euler characteristic of $\sum_{i=0}^{d}(-1)^i\lambda^i(\Omega^1_X)$. The latter was computed via ϵ factors by C-Erez-Pappas-Taylor and C-Pappas-Taylor.

3 The proof depends on the now familiar method of using Chern classes to reduce to working on a dimension 1 subscheme. One then uses results of Taylor and Cassou-Noguès.

4 There are other complexes than the de Rham complex for which one has natural pairings on cohomology of the kind required to define Hermitian discriminants. Serre duality for a vector bundle $E_{\mathbf{Q}}$ on the (smooth) general fiber $X_{\mathbf{Q}}$ of X gives pairings

$$H^i(X_{\mathbf{Q}}, E_{\mathbf{Q}}) \times H^{d-i}(X_{\mathbf{Q}}, E_{\mathbf{Q}}^{dual} \otimes \omega_{X_{\mathbf{Q}}}) \to \mathbf{Q}$$

Thus if one has a symmetric non-degenerate form $E_{\mathbf{Q}} \otimes E_{\mathbf{Q}} \to \omega_{X_{\mathbf{Q}}}$ one can consider Hermitian discriminants for extensions E of $E_{\mathbf{Q}}$ to X.

The Witt group of vector bundles with symmetric bilinear forms having values in a given line bundle has been of studied by various authors, e.g. when $X_{\mathbf{Q}}$ is a Brauer Severi variety (Pumpluen, Szyjewski, Parimala). Brauer Severi varieties over Dedekind rings have been considered by many people (Artin, Van den Bergh, Saltman...).

Applications to the Birch-Swinnerton Dyer Conjecture

Suppose now that $\mathcal{O} = \mathbf{Z}$ and X is as before, so $\dim(X) = 2$.

$V =$ a complex rep. of G, with dual V^*

The associated L-function $L(s, V, Y)$ has conjectural functional equation

$$L(s, V, Y) = \epsilon(Y, V) A(Y, V)^{-s} L(2 - s, V^*, Y)$$

where $\quad A(Y, V) =$ conductor > 0,

$$\epsilon(Y, V) = \prod_{\text{places } v \text{ of } \mathbf{Q}} \epsilon_v(Y, V)$$

$$\epsilon_v(Y, V) = \epsilon_{v,0}(Y, V) \cdot \epsilon(Y_v, V)$$

$Y_v =$ fiber of$Y$ at v if v finite, $\emptyset$ otherwise

Equivariant BSD conjecture: If $d = 1$ and V is absolutely irreducible then

$$\mathrm{ord}_{s=1} L(s, V, Y) = \text{multiplicity of } V \text{ in the } \mathbf{C}G\text{–module } \mathbf{C} \otimes \mathrm{Pic}^0(X)$$

Here $\mathrm{Pic}^0(X) =$ Weil divisor classes on X of degree 0 on generic fiber.

Implication: If $V = V^*$, the functional equation predicts $\epsilon(Y, V) > 0$ if and only if V has even multiplicity in $\mathbf{C} \otimes \mathrm{Pic}^0(X)$.

If V is symplectic, it has a non-degenerate alternating G-invariant form, and it has even multiplicity in $\mathbf{C} \otimes_{\mathbf{Q}} W$ for each $\mathbf{Q}G$-module W.

Take $W = \mathbf{Q} \otimes \mathrm{Pic}^0(X)$.

BSD prediction: $\epsilon(Y, V) > 0$ if V symplectic.

Theorem (Taylor, Pappas, C): Suppose X/H is regular whenever H is a non-cyclic subgroup of order 4 in G. Then

$$\epsilon_v(Y,V) > 0 \quad \text{and} \quad \epsilon(Y,V) > 0$$

for all symplectic representations V of G and all places v of $\mathbf{Q}$.

Comments

- If G is a generalized quaternion group, every subgroup of order 4 in G is cyclic, so the Theorem applies to X.
- It would be interesting if some form of the BSD conjecture also predicted the local ϵ-factors should be positive. We don't know of such a prediction.

- By a Theorem of Serre, every symplectic character of G is a linear combination of characters of the form form

  $$\text{Induction}_{\Gamma}^{G}\ (\text{Inflation}_{\Delta}^{\Gamma}\ \chi)$$

  where Γ is a subgroup of G, Δ is a quotient group of Γ which is either cyclic or a generalized quaternion group, and χ is a symplectic representation of Δ.

- One can't directly reduce to the cyclic and generalized quaternion case because the regularity of X is not preserved.

- Tameness seems essential to the proof. We expect to be able to weaken the hypotheses on $\dim(X)$, Y and G as long as $\dim(X)$ is even. The conclusion of the Theorem need not hold for $\dim(X)$ odd (e.g. for $X = \text{Spec}(O_N)$ and N a number field).

- If V is symplectic, the motive

  $$(H^1(\mathbf{Q} \otimes X) \otimes V)^G$$

  which is most relevant to the Theorem is orthogonal (in the sense of T. Saito). This is because the cup product pairing on the realizations of $H^1(X)$ is alternating. The weight of $(H^1(\mathbf{Q} \otimes X) \otimes V)^G$ is 1.

- **Theorem (T. Saito)** The global ϵ-factor associated to an orthogonal motive of even weight is positive.

  (Positivity is not known in general in weight 1.)

- **Folklore conjecture:** The global ϵ-factors of all orthogonal motives are positive.

- Fröhlich and Queyrut proved the positivity of the ϵ-factors of orthogonal Artin motives (whose weight is 0). Their result and Saito's use no tameness hypotheses. Dirichlet characters of order 2 define orthogonal Artin motives.

- Saito's proof of his theorem uses Deligne's method of identifying the sign of local ϵ-factors with Stieffel-Whitney classes of local orthogonal Galois representations. These local ϵ-factors will not all be positive. But their signs give the local invariants of an element of order two in the Brauer group, so their product is positive.

Outline of the proof

We would like to show under the hypotheses made on X, Y and G,

$$\epsilon_v(Y,V) = \epsilon_{v,0}(Y,V) \cdot \epsilon(Y_v,V) > 0$$

when V is symplectic.

Fix a finite place v, and let $\mathcal{O} = \mathbf{Z}_p$ when $p = p(v)$. Let $Z = X_v^{red}$ be the reduction of the fiber of X over v. We have $S = \{v\}$.

Let $K_0(\mathbf{F}_pG)$ be the Grothendieck group of finitely generated projective modules for $\mathbf{F}_pG$.

Main Idea: Show that both $\epsilon_{v,0}(Y,V)$ and $\epsilon(Y_v,V)$ are determined by an equivariant Euler characteristic $Euler(O_Z)$ in $K_0(\mathbf{F}_pG)$.

Application: Show $\epsilon_{v,0}(Y,V)$ and $\epsilon(Y_v,V)$ have the same sign, implying $\epsilon_v(Y,V) > 0$.

Step 1: Show

$t_p(Euler(O_Z)) = \Delta_f(\epsilon^s_{v,0}(Y))$ in $Ad\,HCl(\mathbf{Z}G)$

under the natural map

$$t_p : K_0(\mathbf{F}_p G) \to Ad\ HCl(\mathbf{Z}G)$$

1A $t_{\mathbf{Z}_p}(d_0(X,G)) = B^{\chi(Y)}\Delta_f(\epsilon^s_{v,0}(Y))$, where $d_0(X,G)$ is the de Rham discriminant

$$d(R\Gamma(X,[O_X \to \omega_X(X^{red}_S)][1]),\langle\ ,\ \rangle)$$

1B Replacing $\omega_X(X^{red}_S)$ by ω_X leads by duality to

$$t_{\mathbf{Z}_p}(d(R\Gamma(X,[O_X \to \omega_X][1]),\langle\ ,\ \rangle)) = B^{\chi(Y)}$$

1C The difference of the left hand sides of #1A and #1B is

$$t_p(Euler(O_Z))$$

by the adjunction formula, while the difference of the right hand sides is $\Delta_f(\epsilon^s_{v,0}(Y))$.

Step 2: Show there is a function $\lambda \in \mathrm{Hom}_\Gamma(R_G^s, \mathbf{Q}^*)$ with only <u>positive</u> values such that

$$t_p(Euler(O_Z)) = \Delta_f(\lambda \cdot \epsilon^s(Y_v)^{-1})$$

where $\epsilon^s(Y_v) \in \mathrm{Hom}_\Gamma(R_G^s, \mathbf{Q}^*)$ is defined via symplectic epsilon constants over the fiber $Y_v = \mathbf{Z}/p \otimes Y$.

Conclusion from Steps 1 and 2: We will get

$$\Delta_f(\epsilon_{v,0}^s(Y) \cdot \epsilon^s(Y_v) \cdot \lambda^{-1}) = 1$$

But by the Theorem of Cassou-Noguès and Taylor, this forces

$$\epsilon_v^s(Y) = \epsilon_{v,0}^s(Y) \cdot \epsilon^s(Y_v) = \lambda$$

to have only positive values. Hence

$$\epsilon_v^s(Y)(V) = \epsilon_v(Y, V) = \epsilon_{v,0}(V) \cdot \epsilon(Y_v, V) > 0$$

for all symplectic V.

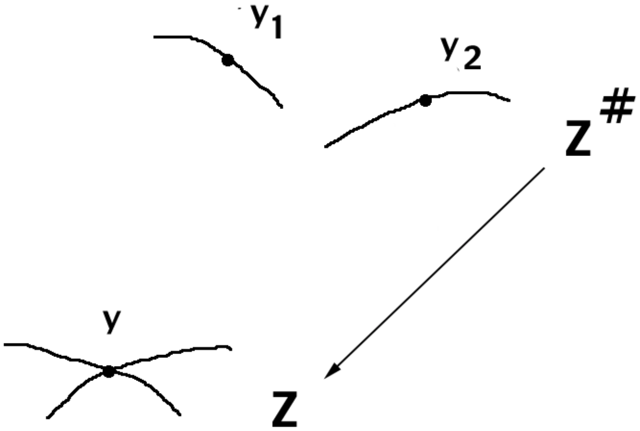
Ingredients for step 2:

2A One shows a λ with only positive values exists for which

$$\frac{t_p(Euler(O_Z))}{t_p(Euler(O_Z^\sharp))} = \Delta_f(\lambda) \quad \text{in} \quad Ad\ HCl(\mathbf{Z}G)$$

where $Z^\sharp$ is the normalization of Z.

The left hand side is the inverse of an Euler characteristic of all the crossing points of components of Z.



2B Step 2 now reduces to showing

$$t_p(Euler(O_Z^\sharp)) = \Delta_f(\epsilon^s(Y_v)^{-1}) \qquad (1)$$

The left hand side is represented by a character function f_{left} into ideles which have component 1 outside p. The character function f_{right} representing the right hand side has values embedded via the "near diagonal" map $\Delta_f : \mathbf{Q}^* \to J(\mathbf{Q})$.

To show the two sides of (1) are equal in $AdHCl(\mathbf{Z}G)$, one has to show

$$f_{left}/f_{right} \quad \text{is in} \quad \mathrm{Det}^s(U(\mathbf{Z}G))$$

i.e. that this ratio is the determinant on symplectic characters of a unit idele. This is a local problem, and formulas of Saito solve this problem at the prime p. One then has to show

$$\epsilon^s(Y_v)^{-1} \in \mathrm{Det}^s((\mathbf{Z}_l G)^*) \quad \text{for all} \quad l \neq p \qquad (2)$$

2C The proof of (2) when the prime l does not divide the order the the generic inertia group of any component of Z follows from work of Saito on epsilon factors and the Lefschetz Riemann-Roch Theorem for curves over a finite field. When l divides the order of a generic inertia group, the situation is more delicate, and one must produce further congruences between epsilon constants. The result now known about this is:

Proposition 4 *One has*

$$\epsilon^s(Y_v)^{-1} \in \mathrm{Det}^s((\mathbf{Z}_l G)^*)$$

if either

A. $p \neq l > 2$, *or*

B. $p \neq l = 2$, *and* X/H *is regular whenever* H *is a non-cyclic subgroup of order* 4 *in* G.