Math104-007 Review on exponential and inverse functions and L'Hospital's rule

1. Exponents and logarithms.

Let a be positive.

- For rational $x = p/q$ the exponent a^x is defined as $a^x = \sqrt[q]{a^p}$. For an arbitrary x the exponent a^x is defined by continuity.
- Basic formulas: $(ab)^x = a^x b^x$, $(a/b)^x = a^x/b^x$, $a^0 = 1$, $a^{-x} = 1/a^x$, $a^{x+y} = a^x a^y$, $a^{x-y} = a^x/a^y$, $a^{xy} = (a^x)^y = (a^y)^x$, $a^{1/x} = \sqrt[x]{a}$. Note that some formulas follow from others.
- Assume that $a \neq 1$, then the logarithm $\log_a(x)$ is defined as the number y such that $a^y = x$. Thus, logarithm is inverse to the exponent and we have the cancellation rules $\log_a(a^y) = y$ and $a^{\log_a x} = x$.
- Basic formulas: $\log_a 1 = 0$, $\log_a(1/x) = -\log_a x$, $\log_a xy = \log_a x + \log_a y$, $\log_a(x/y) = \log_a x - \log_a y$ and the change of base formula

$$\log_a b = \log_c b / \log_c a$$

  for positive a, b, c not equal to 1.

2. Exponential functions.

- For each positive a, a^x is a continuous function defined on the entire real line. It is called exponential function.
- Exponential function is increasing for $a > 1$, decreasing for $a < 1$ and constant for $a = 1$.
- There is an irrational number $e \approx 2.718$ such that $(e^x)' = e^x$. The exponential function e^x is called the natural exponential function. Its integral is given by $\int e^x dx = e^x + C$.
- One can show that $e = \lim_{n\to\infty}(1 + 1/n)^n$.
- For an arbitrary a, one has the following change of base formula

$$a^x = e^{x \ln a}$$

- It follows, in particular, that $(a^x)' = a^x \ln a$ and $\int a^x dx = a^x/\ln a + C$.
- For $a > 1$ the exponential function a^x asymptotically increases faster than any power function, i.e. $\lim_{x\to\infty} a^x/x^n = 0$ for any n. Similarly, for $a < 1$ the exponential function a^x asymptotically decreases faster than any power function, i.e. $\lim_{x\to\infty} a^x/x^n = 0$ for any n (taking $n = -1000$, for example, we obtain that $\lim_{x\to\infty} x^{1000}a^x = 0$).
- For $x \to -\infty$ the situation is opposite: a^x tends to ∞ for $a < 1$ and a^x tends to 0 for $a > 1$.
- The behavior of a function $x^n a^x$ as x tends to $\pm\infty$ is similar to the behavior of a^x, i.e. a^x is the dominant factor.

3. Inverse functions.

- A function $f(x)$ is one-to-one if $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$.
- If $f(x)$ is one-to-one with domain A and range B, then its inverse $g(x) = f^{-1}(x)$ is defined by $f(x) = y$ if and only if $f^{-1}(y) = x$.

It has domain B and range A and satisfies the cancellation rules $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$.

- The graph of f^{-1} is obtained from the graph of f by reflection about the line $y = x$. In other words, we switch the coordinate axis.
- Examples: $f(x) = x + c, g(x) = x - c$; $f(x) = x^n, g(x) = \sqrt[n]{x} = x^{1/n}$; $f(x) = a^x, g(x) = \log_a(x)$.
- If $f(x)$ is continuous then $f^{-1}(x)$ is continuous.
- If $f(a) = b$ and f is differentiable at a, then $g(x) = f^{-1}(x)$ is differentiable at b and $f'(a)g'(b) = 1$.

4. Logarithmic functions.
Let $a \neq 1$ be a positive number.

- The logarithmic function $\log_a x$ is defined for $x > 0$. It is increasing for $a > 1$ and decreasing for $a < 1$.
- The natural logarithm function is defined as $\ln x = \log_e x$.
- Any logarithmic function is proportional to the natural logarithm by the change of base formula

$$\log_a(x) = \frac{\ln x}{\ln a}$$

- Asymptotically, $\ln x$ growthes slower than any increasing power function, i.e. $\lim_{x\to\infty} \ln x / x^n = 0$ for any $n > 1$. So, the behavior of a function $x^n \ln(x)^m$ as x tends to ∞ is similar to the behavior of x^n, i.e. x^n is the dominant factor.
- The formula for the derivative of the inverse function implies that

$$(\ln x)' = \frac{1}{x}$$

and $(\log_a(x)' = x^{-1} \ln a$

- In particular, we obtain the following basic integration formula

$$\int \frac{dx}{x} = \ln|x| + C$$

- One could guess the formula $\int \ln x dx = x \ln x - x + C$, but the best way to integrate $\ln x$ is to integrate by parts (chapter 8).

5. Logarithmic derivative.

- Logarithm can be a useful tool to differentiate products and functions involving powers, e.g. x^x.
- The basic formula is

$$\frac{d \ln f(x)}{dx} = \frac{f'(x)}{f(x)}$$

hence $f'(x) = f(x)\,(\ln f(x))'$

- For example, for $f(x) = x^x$ we have $f'(x) = f(x)(\ln(x^x))' = x^x(x \ln x)' = x^x(\ln x + 1)$.

- Another way to differentiate $f(x)$ is to notice that $x^x = e^{x\ln x}$ by the change of base formula, hence
$$f'(x) = e^{x\ln x}(x\ln x)' = e^{x\ln x}(\ln x + 1) = x^x(\ln x + 1)$$

6. Inverse trigonometric functions

- $\sin x$ is not one-to-one on the real line, but it is one-to-one on the interval $[-\pi/2, \pi/2]$, and we define $\sin^{-1}(x)$ to be its inverse on this interval, i.e. $\sin^{-1} x = y$ if and only if $\sin y = x$ and $-\pi/2 \leq y \leq \pi/2$.
- The domain of $\sin^{-1}(x)$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$.
- By the cancellation rule, $\sin(\sin^{-1}(x)) = x$ whenever the left side is defined, i.e. $x \in [-1, 1]$. Similarly, $\sin^{-1}(\sin(x)) = x$ for any $x \in [-\pi/2, \pi/2]$.
- Caution: $\sin^{-1}(\sin(x)) \neq x$ for x not in the interval $[-\pi/2, \pi/2]$.
- $\cos^{-1}(x)$ is the inverse of $\cos(x)$ on the interval $[0, \pi]$, thus $0 \leq \cos^{-1}(x) \leq \pi$.
- One has $\sin^{-1} x + \cos^{-1} x = \pi/2$.
- $\tan^{-1}(x)$ is the inverse of $\tan(x)$ on the interval $(-\pi/2; \pi/2)$. Note that $\tan(x)$ is not defined at the endpoints.
- The domain of $\tan^{-1}(x)$ is the entire line $(-\infty, \infty)$ and the range is $(-\pi/2, \pi/2)$.
- By the cancellation rule, $\tan(\tan^{-1}(x)) = x$ for any x and $\tan^{-1}(\tan(x)) = x$ for any $x \in (-\pi/2, \pi/2)$ (and only for those x's!).
- The derivatives are
$$\frac{d\sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}}, \frac{d\cos^{-1} x}{dx} = \frac{-1}{\sqrt{1-x^2}}, \frac{d\tan^{-1} x}{dx} = \frac{1}{1+x^2}$$
- As a consequence we obtain the following basic (or table) integrals which should be memorized:
$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$
$$\int \frac{dx}{x^2+1} = \tan^{-1} x + C$$
and, more generally,
$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Example of two integrals that can be found by use of inverse functions:

- $$\int \frac{dx}{x^2+1} = \tan^{-1} x + C$$
- $$\int \frac{xdx}{x^2+1} = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{\ln|u|}{2} + C = \frac{\ln(x^2+1)}{2} + C$$

Often, one has to simplify expressions like $f(\sin^{-1} x)$ and $f(\tan^{-1} x)$, where $f(x)$ is a trigonometric function. It can be done using the cancellation rules and the two basic identities

$$\sin^2 x + \cos^2 x = 1, \tan^2 x + 1 = \sec^2 x$$

- To simplify $f(\sin^{-1} x)$, assume that $\alpha = \sin^{-1} x \in [-\pi/2, \pi/2]$ is an unknown angle.
- Then, $\sin\alpha = x$ by the cancellation rule.
- Hence $\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - x^2}$ (because $\cos x$ is non-negative on $[-\pi/2, \pi/2]$).
- Using $\cos\alpha$ and $\sin\alpha$ we can express any trigonometric expression $f(\alpha)$, e.g. $\sec\alpha = 1/\cos\alpha = 1/\sqrt{1-x^2}$, $\tan\alpha = \sin\alpha/\cos\alpha = x/\sqrt{1-x^2}$, etc.
- To simplify $f(\tan^{-1} x)$, assume that $\alpha = \tan^{-1} x \in (-\pi/2, \pi/2)$ is an unknown angle.
- Then, $\tan\alpha = x$ by the cancellation rule.
- Hence $\sec\alpha = \sqrt{1 + \tan^2\alpha} = \sqrt{1 + x^2}$ (because $\sec x$ is non-negative on $(-\pi/2, \pi/2)$).
- Using $\sec\alpha$ and $\tan\alpha$ we can express any trigonometric expression $f(\alpha)$, e.g. $\cos\alpha = 1/\sec\alpha = 1/\sqrt{1+x^2}$, $\sin\alpha = \tan\alpha/\sec\alpha = x/\sqrt{1+x^2}$, etc.

7. L'Hospital's rule.

Consider a limit of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ where a is a number, or a symbol b^+ or b^-, or $\pm\infty$ and assume that $\lim_{x\to a} f(x) = \pm \lim_{x\to a} g(x)$ is either 0 or $\pm\infty$. This information does not determine the value of the original limit, so one says that the limit is of indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. L'Hospital's rule states that in this situation

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

- Caution: always check that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ is either 0 or ∞. The rule is wrong without this assumption.
- The rule can be used iteratively, but try to simplify the expression after each usage of the rule.
- Example:

$$\lim_{x\to 0} \frac{e^x - x}{x^3} = \lim_{x\to 0} \frac{e^x - 1}{3x^2} = \lim_{x\to 0} \frac{e^x}{6x}$$

  since $\lim_{x\to 0} e^x = 1$, the latter expression tends to infinity. If one misses that $\lim_{x\to 0} e^x \neq 0$ and applies L'Hospital's rule once again, then a wrong answer is obtained.

8. Indeterminate forms.

- There are seven indeterminate forms of limits: $0/0, \infty/\infty, 0\infty, \infty - \infty, 1^\infty, \infty^0, 0^0$.

- The limits of the form 0/0 and ∞/∞ can be treated by L'Hospital's rule.
- The limit $\lim_{x\to a} f(x)g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$ can be rewritten as $\lim_{x\to a} \frac{f(x)}{1/g(x)}$ which is of the form 0/0.
- The limit $\lim_{x\to a}(f(x)-g(x))$ where $f(x), g(x) \to \infty$ sometimes can be converted to the above forms. The formula $f - g = \frac{f^2-g^2}{f+g}$ might be useful.
- The limits of the form $\lim_{x\to a} f(x)^{g(x)}$ should be rewritten using the change of base formula as

$$\lim_{x\to a} e^{g(x)\ln(f(x))}$$

By continuity of e^x it suffices to find $L = \lim_{x\to a} g(x)\ln(f(x))$ and then the original limit equals to e^L (where $e^\infty = \infty$ and $e^{-\infty} = 0$).

9. Hyperbolic functions.

Hyperbolic functions are not in the exam syllabus. However, they are in the general course syllabus and will be used in more advanced calculus courses.

- Hyperbolic functions are defined using the natural exponential function. Surprisingly, they behave very similarly to usual trigonometric functions.
- The definitions are

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}, \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

- The following formulas are very similar to their trigonometric analogs $\cosh(x)^2 - \sinh(x)^2 = 1$, $\cosh'(x) = \sinh(x)$, $\sinh'(x) = \cosh(x)$, $\tanh'(x) = 1/\cosh(x)^2$
- The inverse hyperbolic functions can be computed using logarithms: $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$ and $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$.
- Their derivatives are

$$\frac{d\sinh^{-1} x}{dx} = \frac{1}{\sqrt{1+x^2}}, \frac{d\cosh^{-1} x}{dx} = \frac{1}{\sqrt{x^2-1}}, \frac{d\tanh^{-1} x}{dx} = \frac{1}{1-x^2}$$