Math104-007; Review on sequences, series and power series.

1. Sequences.

- A sequence $\{a_n\}_{n=1}^{\infty}$ is a set $a_1, a_2, a_3, \ldots$ of real numbers. Usually, we start the numeration with $n = 1$, but sometimes it is convenient to start it, for example, with $n = 0$.
- A sequence can be given by a formula, e.g. $a_n = \ln(n^2 + 1)$, or, more generally, $a_n = f(n)$ for a function $f(x)$. On the other hand, a sequence can be given by recursion, e.g. the Fibonacci sequence defined as $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$, so $a_3 = 1 + 1 = 2, a_4 = 2 + 1 = 3, a_5 = 3 + 2 = 5$, etc.
- A sequence $\{a_n\}$ has a limit L if we can make a_n arbitrary close to L by choosing n large enough.
- If L exists, then we write

  $$\lim_{n\to\infty} a_n = L$$

  and say that the sequence converges, otherwise we say that the sequence diverges.
- A particular example of divergence is obtained then a_n gets larger then any number M for sufficiently large n. Then we write that $\lim_{n\to\infty} a_n = \infty$.
- Example: $1, -1, 1, -1, 1, \ldots$ diverges but do not tend to infinity. The latter sequence is given by the formula $a_n = (-1)^{n-1}$. Caution: $f(x) = (-1)^x$ is not defined as a function.
- Convergence criterion: if $f(x)$ is a function such that $\lim_{x\to\infty} f(x) = L$ exists then $\lim_{n\to\infty} a_n$ exists and equals to L.
- Caution: even when $\lim_{x\to\infty} f(x)$ does not exist it can happen that $\lim_{n\to\infty} a_n$ converges. For example, $\lim_{x\to\infty} \sin \pi x$ does not exist, but $\lim_{n\to\infty} \sin \pi n = 0$ because each term of the latter sequence is zero.
- There are limit laws similar to the usual limits laws for functions:

  $$\lim_{n\to\infty} Ca_n = C \lim_{n\to\infty} a_n$$

  $$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

  $$\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$$

  et cetera. Here the equalities mean that if the right hand side is defined (i.e. both limits exist), then the left hand side is defined and the equality holds.
- Caution: it can happen that both limits on the right hand side do not exist, but the left hand side is defined.
- The Squeeze Theorem: if $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences with $a_n \le b_n \le c_n$ for any n, and the sequences $\{a_n\}$ and $\{c_n\}$ converge to the same limit L, then so does the sequence $\{b_n\}$.
- In particular, if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$ because $-|a_n| \le a_n \le |a_n|$ for any n.

- A sequence $\{a_n\}$ increases if $a_{n+1} \geq a_n$ for any n. A sequence is bounded from above if $a_n < M$ for any n and some fixed number M.
- Monotonic Sequence Theorem: any increasing bounded from above sequence converges.
- The theorem is often used to find the limit of recurrent sequences.
- Similarly, any decreasing bounded from below sequence converges.

2. **Series.**
   - A series is an infinite sum $a_1 + a_2 + a_3 + \dots$ of real numbers, which can be also written as $\sum_{n=1}^{\infty} a_n$. Similarly to sequences, series can start with another index, e.g. $\sum_{n=0}^{\infty} a_n$.
   - The partial sums of $\sum_{n=1}^{\infty} a_n$ are the numbers $s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$.
   - If the sequence $\{s_n\}$ of partial sums has a finite limit s, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges and s is its sum. We write $\sum_{n=1}^{\infty} a_n = s$. If the sequence $\{s_n\}$ diverges then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.
   - Example: assume $a \neq 0$, then the geometric series $a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$ diverges for $|r| \geq 1$. For $|r| < 1$ the series converges and
   $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$
   - Finitely many terms do not affect convergence of a series, so $a_1 + a_2 + \dots$ converges if and only if $a_{100} + a_{101} + a_{102} + \dots$ converges.
   - Test for divergence: if $\lim_{n\to\infty} a_n \neq 0$ or the limit does not exists, then the series $\sum a_n$ diverges. In particular, if the series converges, then necessarily $\lim_{n\to\infty} a_n = 0$
   - Caution: the converse is not true in general. For example, the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges, though its terms $\frac{1}{n}$ tend to zero.
   - If $\sum a_n$ and $\sum b_n$ converge and C is a number then
   $$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n)$$
   $$\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n - b_n)$$
   $$\sum_{n=1}^{\infty} Ca_n = C \sum_{n=1}^{\infty} a_n$$

3. Tests for convergence/divergence.

Integral test.

- If $f(x)$ is continuous positive decreasing function on $[1,\infty)$, then the series $\sum_{n=1}^{\infty} f(n)$ converges/diverges if and only if the improper integral $\int_1^{\infty} f(x)dx$ does so.
- Example: the p-series
$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$
converges for $p > 1$ and diverges for $p \leq 1$ because so does the integral $\int_1^{\infty} \frac{dx}{x^p}$.
- Example: more generally, the series $\sum_{n=1}^{\infty} \frac{1}{n^p(\ln n)^q}$ converges for $p > 1$ or $p = 1, q > 1$.
- Remainder: if the series $\sum_{n=1}^{\infty} f(n)$ as above converges then the n-th remainder $R_n = s - s_n = a_{n+1} + a_{n+2} + \dots = \sum_{i=n+1}^{\infty} a_i$ can be estimated as
$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

Comparison test.

- If $0 \leq a_n \leq b_n$ for any n then: (i) if $\sum b_n$ converges then $\sum a_n$ converges; (ii) if $\sum a_n$ diverges then $\sum b_n$ diverges.
- Limit comparison: if $a_n \geq 0, b_n \geq 0$ for any n and $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ for a finite non-zero C then: i) if $\sum a_n$ converges if and only if $\sum b_n$ converges; (ii) $\sum a_n$ diverges if and only if $\sum b_n$ diverges.
- It is worth to remember the following estimates
$$(\ln n)^a \ll n^b \ll r^n$$
for $a, b > 0, r > 1$. The latter means that $\lim_{n\to\infty} (\ln n)^a / n^b = 0$ and $\lim_{n\to\infty} n^b / r^n = 0$, i.e. any power of $\log n$ grows slower than polynomials in n, and polynomials grow slower than exponents.

Alternating test.

- An alternating test $a_1 - a_2 + a_3 - a_4 + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ with $a_n > 0$ and $a_1 \geq a_2 \geq a_3 \geq \dots$ converges if and only if $\lim_{n\to\infty} a_n = 0$.
- Caution: do not forget to check that a_n tend to zero!
- Example: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots$ converges, but $\sum_{n=1}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - \dots$ diverges.
- The remainder can be estimated as $|R_n| \leq a_n$.
- Similar statements hold for an alternating series $-a_1 + a_2 - a_3 + \dots$.

4. Absolute and conditional convergence.

- A series $a_1 + a_2 + a_3 + \dots$ absolutely converges if the series $|a_1| + |a_2| + |a_3| + \dots$ converges.
- If a series absolutely converges, then it converges.

- A series converges conditionally if it converges but not absolutely converges.
- Example: $\sum \frac{(-1)^n}{n}$ converges conditionally by the alternating test and p-series test, $\sum \frac{(-1)^n}{n^2}$ converges absolutely by the p-series test.

The ratio test: assume that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$, then

- If $L < 1$ then $\sum a_n$ absolutely converges.
- If $L > 1$ then $\sum a_n$ diverges.
- If $L = 1$ then the test is not informative in the sense that the series can diverge, conditionally converge or absolutely converge.

The root test: assume that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, then

- If $L < 1$ then $\sum a_n$ absolutely converges.
- If $L > 1$ then $\sum a_n$ diverges.
- If $L = 1$ then the test is not informative in the sense that the series can diverge, conditionally converge or absolutely converge.

5. Strategy for testing series.

I recommend to test convergence/divergence of a series $\sum_{n=1}^{\infty} a_n$ using the following strategy.

- Try to recognize the series, e.g. geometric series or p-series.
- Try to check if $\lim_{n\to\infty} a_n = 0$. If the limit is not zero (or does not exist) then the series diverges.
- If the series is alternating, then use the alternating test.
- If a_n is an n-th power, then try the root test.
- If a_n involves products, factorials and powers, then try the ratio test.
- If a_n can be approximated by a simpler term b_n, then try (limit) comparison with respect to b_n. For example,

$$\frac{3^n + n^2}{(\ln n)^3 + 5^n} \approx \frac{3^n}{5^n} = \left(\frac{3}{5}\right)^n$$

- In order to find a good approximation you can try to remove all non-dominant terms. For example, in the above example we used that $n^2 \ll 3^n$ and $(\ln n)^3 \ll 5^n$.
- If $a_n = f(n)$ for a nice function $f(x)$, then you can try to use the integral test (though it is a relatively rare bird).

6. Power series.

- A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

- More generally, a power series with center at a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

- A power series is a bunch of usual series: for any value of x we get a usual series by substituting that value.
- For any power series with center at a there exists R such that the series absolutely converges for $|x-a| < R$ and diverges for $|x-a| > R$. This R is called the radius of convergence of the power series. It can be a non-negative number (including zero) or infinity.
- For the critical points $x = a \pm R$ anything can happen: the resulting series can diverge, absolutely converge or conditionally converge.
- To find R use the ratio test, i.e. find R such that
$$\lim_{n\to\infty}\left|\frac{c_n R^n}{c_{n+1}R^{n+1}}\right| = 1$$
In other words,
$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$
- The exact interval of convergence can be $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$ or $[a-R, a+R]$. To find it we first find R and then test convergence at the end-points $a \pm R$. The latter always requires a more subtle test than the ratio test.
- Examples: the geometric power series $\sum_{n=0}^{\infty} x^n$ has $R = 1$ (we know that the geometric series absolutely converges for $|x| < 1$ and diverges for other x's); for $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ one has $R = \infty$; for $\sum_{n=0}^{\infty} n!x^n$ one has $R = 0$.

7. Power series as functions.

- If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R, then for each $x \in (a-R, a+R)$ the series converges and its sum defines a value $f(x)$. So, the power series defines a function $f(x)$ on the interval of convergence $(a-R, a+R)$ which is called the sum of the power series (sometimes the endpoints $x = a \pm R$ can also be included).
- Example: the geometric series
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad x \in (-1, 1)$$
- One can differentiate and integrate power series term-wise:

$$(c_0+c_1(x-a)+c_2(x-a)^2+\dots)' = c_1+2c_2(x-a)+3c_3(x-a)^2+\cdots = \sum_{n=0}^{\infty}(n+1)c_{n+1}(x-a)^n$$

$$\int(c_0+c_1(x-a)+c_2(x-a)^2+\dots)dx = C+c_0(x-a)+\frac{c_1}{2}(x-a)^2+\frac{c_2}{3}(x-a)^3+\cdots = C+\sum_{n=1}^{\infty}\frac{c_{n-1}}{n}(x-a)^n$$

- Sometimes, one chooses the particular antiderivative with $C = 0$. It is done by integrating from a to x:

$$\int_a^x (c_0+c_1(t-a)+c_2(t-a)^2+\dots)dt = c_0(x-a)+\frac{c_1}{2}(x-a)^2+\frac{c_2}{3}(x-a)^3+\cdots = \sum_{n=1}^{\infty}\frac{c_{n-1}}{n}(x-a)^n$$

- The new power series (both derivative and integral) have the same radius of convergence (though the convergence at the endpoints can be affected).

8. Taylor and Maclaurin series.

- If $f(x)$ has a power series representation $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for $|x-a| < R$ with $R > 0$, then necessarily $c_n = \frac{f^n(a)}{n!}$. Thus,

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f'''(a)}{6}(x-a)^3+\cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- The power series

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f'''(a)}{6}(x-a)^3+\cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

  is called the Taylor power series of $f(x)$ at $x = a$ (or around a) regardless to the question of its convergence.
- Caution: it does not need to converge in general, and even worse, it can converge to something different from $f(x)$.
- The important particular case is obtained for $a = 0$, it is called the Maclaurin series of $f(x)$

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

- The Taylor polynomial of order n is obtained by cutting the Taylor series after the n-th power term:

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- Remainder: if $|f^{n+1}(x)| \leq M$ for $|x-a| < R$ then the remainder $R_n(x) = f(x) - P_n(x)$ of the Taylor series around a satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

  for $|x-a| < R$.
- Estimating the remainder often gives a tool to prove that the Taylor series of $f(x)$ around a converges to $f(x)$ on $(a-R, a+R)$.

9. Standard Taylor series.

Here is a list of standard Taylor series which converge to their functions on the interval of convergence. We provide also the radius of convergence and the exact interval of convergence. Also, we use the convention $0! = 1$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n,\ x \in (-1,1),\ R = 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ x \in (-\infty,\infty),\ R = \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},\ x \in (-\infty,\infty),\ R = \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},\ x \in (-\infty,\infty),\ R = \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n},\ x \in (-1,1],\ R = 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1},\ x \in [-1,1],\ R = 1$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n,\ x \in (-1,1),\ R = 1$$

$$(1+x)^k = 1 + \frac{k}{1}x + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n,\ R = 1$$

Notice that the formulas for $\frac{1}{1-x}$ and $\frac{1}{(1-x)^2}$ can be obtained from the formula for $(1+y)^k$ by taking $y = -x, k = -1$ and $y = -x, k = -2$.

- Power series representations of other series can often be obtained from the standard ones by term-by-term integration/differentiation, or applying arithmetic operations and substitution to the standard power series.
- Example:

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots$$

- The same computation in the compact form is done as follows

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

- Example: the "impossible" integral

  $$\int e^{x^2} dx$$

  can be easily expressed as a power series

$$\int \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \right) dx = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} = C + x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \dots$$

- Example: another "impossible" integration is

$$\int \frac{\sin x}{x} dx = \int \frac{x - x^3/3! + x^5/5! - \dots}{x} dx = \int \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \right) dx =$$

$$C + x - \frac{x^3}{18} + \frac{x^5}{5 \cdot 5!} - \dots$$

  In particular,

  $$\int_0^x \frac{\sin t}{t} dt = x - \frac{x^3}{18} + \frac{x^5}{5 \cdot 5!} - \dots$$

- Exercise: do the same computation in the compact form to show that

  $$\int_0^x \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1) \cdot (2n+1)!}$$