Math 114 Sec 002, Spring 2007
Midterm 1

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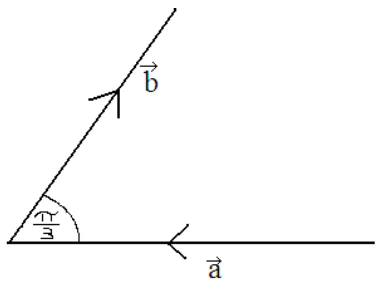
Solutions

1. You know that the equation of the plane P_1 is $ax + by + cz = d$, and that the plane P_2 contains the three points A, B and C. Explain briefly how you would check whether the planes P_1 and P_2 are parallel.

   Solution: Normal to P_1 is $\vec{n}_1 =< a, b, c >$; to find normal $\vec{n}_2$ to P_2, compute the cross product of two vectors in P_2, for example the vectors $\vec{AB}$ and $\vec{AC}$. The two planes are parallel exactly when $\vec{n}_1$ is parallel to $\vec{n}_2$. So to see if the planes are parallel, check whether $\vec{n}_1$ is a scalar multiple of $\vec{n}_2$.

2. When $\vec{a}$ and $\vec{b}$ are represented by two line segments with the terminal point of $\vec{a}$ coinciding with the initial point of $\vec{b}$, the smaller of the two angles between the two line segments is $\frac{\pi}{3}$ (see the picture below). Find $\vec{a} \cdot \vec{b}$ if $|\vec{a}| = 7$ and $|\vec{b}| = 4$. (Give your answer as a number; e.g., "10" instead of "$10 \sin(\pi/2)$".)



   Solution: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ where θ is angle between two vectors *when they are represented by line segments that share a common start point*. So in this case $\theta = 2\pi/3$ (not $\pi/3$) and so $\vec{a} \cdot \vec{b} = 7.4 \cos 2\pi/3 = -14$.

3. One of the pictures below shows the graph of the quadric surface A whose equation is $x^2 - y^2 + z^2 - 1 = 0$, another shows the graph of the surface B with equation $x^2 - y^2 + z^2 + 1 = 0$, and another shows the graph of the surface C with equation $x^2 - y - z^2 = 0$. Write the letters "A", "B" and "C" beside the correct graphs. Give a brief justification for your choices.

   Solution: A is Figure 2; traces in y direction are circles for all y, and trace at $y = 0$ is circle with non-zero radius. B is Figure 3; traces in y direction are circles for all $y \geq 1$ and $y \leq -1$; for y between -1 and 1 the traces are empty. C is Figure 4; trace at $z = 0$ is a parabola.

4. Show that if three non-zero and non-parallel vectors $\vec{a}, \vec{b}$ and $\vec{c}$ all lie in the same plane, then

$$\vec{a}.(\vec{b} \times \vec{c}) = 0.$$

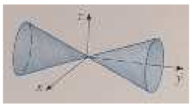


Figure 1: Quadric Surface 1

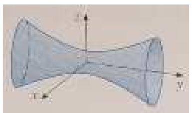


Figure 2: Quadric Surface 2

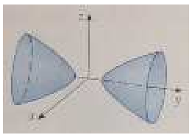


Figure 3: Quadric Surface 3

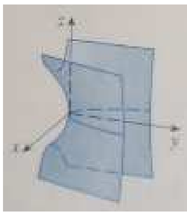


Figure 4: Quadric Surface 4

Solution: $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ and so is perpendicular to the plane spanned by $\vec{b}$ and $\vec{c}$. That means it's perpendicular to $\vec{a}$ and so $\vec{a}.(\vec{b} \times \vec{c}) = 0$.

5. Show that the curve with parametric equations

$$x = 2\cos 2t \quad y = 2\sin 4t \quad z = 2\sin 2t$$

lies on the intersection of the surfaces $y = xz$ and $x^2 + z^2 = 4$.

Solution: We just check that points on the curve are on both surfaces, by checking that

$$xz = 2\cos 2t \times 2\sin 2t = 2(2\cos t \sin t) = 2\sin 4t = y$$

and

$$x^2 + z^2 = (2\cos 2t)^2 + (2\sin 2t)^2 = 4(\cos^2 2t + \sin^2 2t) = 4.$$

6. A surface is given in spherical coordinates by the equation

$$\rho = 2\sin\phi\cos\theta + 2\sin\phi\sin\theta + 2\cos\phi.$$

Convert the equation to rectangular coordinates and describe the surface.

Solution: $\rho = 2\sin\phi\cos\theta + 2\sin\phi\sin\theta + 2\cos\phi$, so multiplying both sides by ρ get

$$\rho^2 = 2\rho\sin\phi\cos\theta + 2\rho\sin\phi\sin\theta + 2\rho\cos\phi$$

or

$$x^2 + y^2 + z^2 = 2x + 2y + 2z$$

or

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 3,$$

which is a sphere of radius $\sqrt{3}$ with center $(1, 1, 1)$.

7. Find the arc length function of the curve with parametrization $\vec{r}(t) = \langle e^t + 2, e^{-t} - 2, \sqrt{2}t - 2\rangle$ starting at $t = 1$. If your answer involves an integral, you should evaluate it.

   Solution:

$$\begin{aligned} s(t) &= \int_1^t |\vec{r}'(x)|\, dx \\ &= \int_1^t \sqrt{(e^x)^2 + (-e^{-x})^2 + (\sqrt{2})^2}\, dx \\ &= \int_1^t \sqrt{e^{2x} + 2 + e^{-2x}}\, dx \\ &= \int_1^t \sqrt{(e^x + e^{-x})^2}\, dx \\ &= \int_1^t (e^x + e^{-x})\, dx \\ &= \left[e^x - e^{-x}\right]_1^t \\ &= e^t - e^{-t} - e + e^{-1}. \end{aligned}$$

8. Projectiles A and B are launched from the same spot on the ground. A has initial speed 10 meters per second and angle of elevation $\pi/6$. B has initial speed 6 meters per second and angle of elevation $\pi/3$. Which of A, B has the greater vertical range (i.e., reaches the highest point above the ground)? Justify your answer.

   Solution: The equation of motion for a particle launched from the origin with initial speed v_0 and angle of elevation α is

$$\vec{r}(t) =< (v_0 \cos\alpha)t, (v_0 \sin\alpha)t - gt^2/2 > .$$

   The highest point is reached when the derivative of the y coordinate above is zero, that is when $t = (v_0 \sin\alpha)/g$. At this time the height reached is $(v_0 \sin\alpha)^2/(2g)$. So projectile A reaches maximum height $(10 \sin\pi/6)^2/(2g) = 25/(2g)$ and projectile B reaches maximum height $(6 \sin\pi/3)^2/(2g) = 27/(2g)$. Whatever the value of g, B has the greater vertical range.

9. A particle of mass $1kg$ is acted on by the force $\langle 6t, -2t + 1, e^{t/2}\rangle$ at time t. The particle starts from rest (zero velocity) at the point $(1, 1, -1)$ at the time $t = 0$. What point is at at time $t = 1$?

   Solution: By Newton, the acceleration of the particle satisfies

$$\vec{a}(t) = \langle 6t, -2t + 1, e^{t/2}\rangle.$$

   Integrating and using the initial condition $\vec{v}(0) =< 0, 0, 0 >$ get

$$\vec{v}(t) = \langle 3t^2, -t^2 + t, 2e^{t/2} - 2\rangle.$$

Integrating again and using the initial condition $\vec{r}(0) =< 1, 1, -1 >$ get

$$\vec{r}(t) = \langle t^3 + 1, -t^3/3 + t^2/2 + 1, 4e^{t/2} - 2t - 5\rangle.$$

So at $t = 1$ the particle is at the point $(2, 7/6, 4\sqrt{e} - 7)$.

10. Let $\vec{w} =< x, y >$ with $x \geq 0, y \geq 0$. Find the length of the projection of $\vec{w}$ onto the vector $\vec{v} = \vec{i}+2\vec{j}$.

Solution: The projection vector is

$$\text{proj}_{\vec{v}}\vec{w} = \left(\frac{\vec{w}.\vec{v}}{\vec{v}.\vec{v}}\right)\vec{v} = \frac{x + 2y}{5} < 1, 2 >$$

which has length $\frac{x+2y}{\sqrt{5}}$.