

Math114; Comments to the solution to the first midterm

The goal of these comments is to describe few typical mistakes, and to point out few alternative solutions. In the solved version of the midterm, we chose short and not computational solutions, but few of them were rather tricky (marked with asterisks). Here we suggest more straightforward, but more laborious ones.

Problems 1 - 4 are rather standard. A typical mistake in 1. was to take $(2, -5, 7)$ instead of $\mathbf{n} = (2, -5, -1)$ as a normal vector to the plane $z = 2x - 5y + 7$.

5. (i) The most typical mistake was to answer c) in the first part. Indeed, c) defines a portion of a cylinder, but it is an unbounded from below portion.

(ii) It was very useful to remember that a single equation (in any coordinate system) *usually* defines a surface, two equations define a curve, and inequalities just cut off parts of the same dimension. So, a core is given by inequalities, a wire is given by two equations and inequalities, etc. (Caution: sometimes this rule does not apply, e.g. $x^2 + y^2 = 0$ is the z -axis).

6. The problem caused to many confusions. The correct facts are as follows: (i) the distance between $\mathbf{r}(1)$ and $\mathbf{r}(3)$; (ii) the length of the curve traced by $\mathbf{r}(t)$ from $t = 1$ to $t = 3$; (iii) the vector between $\mathbf{r}(1)$ and $\mathbf{r}(3)$ may be found as

$$(i) \left| \int_1^3 \mathbf{r}'(t) dt \right|; \quad (ii) \int_1^3 |\mathbf{r}'(t)| dt; \quad (iii) \int_1^3 \mathbf{r}'(t) dt$$

7. (i) A standard mistake was to answer C) because the magnitude of $\mathbf{v} \times \mathbf{x}$ is maximal for $\alpha = 90^\circ$. This would be true if the length of \mathbf{x} was fixed (in addition to the length of \mathbf{v}).

(ii) A possible solution is to use that $|\mathbf{v} \times \mathbf{x}|$ is the area of the parallelogram defined by \mathbf{v} and \mathbf{x} . Since \mathbf{v} is given, this area is maximized by choosing \mathbf{x} to be of maximal possible height over the line of \mathbf{v} .

8. A straightforward approach is to notice that $t = 0$ and to compute $\mathbf{r}'(0)$. Since the latter is $\mathbf{0}$, the curve is not smooth with respect to this parameterization, so we have to reparameterize it. If we miss the substitution $x = t^3$, then the natural way is to use the parameterization by arc length:

$$\begin{aligned} s(t) &= \int_0^t |\mathbf{r}'(z)| dz = \int_0^t \sqrt{(-3z^2 \sin z^3)^2 + (3z^2 \cos z^3)^2 + (3z^2)^2} dz = \\ &= \int_0^t 3z^2 \sqrt{(\sin z^3)^2 + (\cos z^3)^2 + 1} dz = \int_0^t 3z^2 \sqrt{2} dz = \sqrt{2} t^3 \end{aligned}$$

Inverting $s(t) = \sqrt{2} t^3$ we get $t = 2^{-\frac{1}{6}} s^{\frac{1}{3}}$, hence

$$\mathbf{r}(s) = \left(\cos \left(\frac{s}{\sqrt{2}} \right), \sin \left(\frac{s}{\sqrt{2}} \right), \frac{s}{\sqrt{2}} \right)$$

is the arc length parameterization. It remains to use that

$$T(0) = \left. \frac{d\mathbf{r}}{ds} \right|_{s=0} = \left(\frac{-\sin 0}{\sqrt{2}}, \frac{\cos 0}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \frac{\mathbf{j}}{\sqrt{2}} + \frac{\mathbf{k}}{\sqrt{2}}$$

9. (i) The point corresponds to $t = 1$. The osculating plane at time $t = 1$ is defined by the point $\mathbf{r}(1)$ and two vectors $\mathbf{T}(1)$ and $\mathbf{N}(1)$ parallel to the plane. In principle, one can find $\mathbf{T} = \mathbf{T}(1)$ and $\mathbf{N} = \mathbf{N}(1)$, then $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is a normal vector to the osculating plane. If $\mathbf{B} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then the linear equation of the plane is of the form $ax + by + cz + d = 0$, and d is found from the fact that $\mathbf{r}(1)$ is in the plane.

(ii) We will see that it may be difficult to compute \mathbf{N} in our case, so one can wonder if one can compute other vectors in the osculating plane in order to use them instead of \mathbf{T} and \mathbf{N} . First, $\mathbf{r}'(1)$ is a tangent vector parallel to the unit tangent \mathbf{T} , so we can use $\mathbf{r}'(1)$ instead of \mathbf{T} . Now, the question is how to construct another vector \mathbf{v} in the osculating plane so that \mathbf{v} is not parallel to the tangent vector. The trick (worth two asterisks) is to recall that the acceleration $\mathbf{r}''(1)$ lies in the osculating plane. Then a normal vector can be found as $\mathbf{r}'(1) \times \mathbf{r}''(1)$ and the rest is easy. This leads to a short tricky solution (see the solution for the midterm for details).

(iii) Now, let us describe a more typical solution. First, $\mathbf{r}'(t) = e^t\mathbf{i} + \mathbf{j} - t^{-2}\mathbf{k}$, hence $\mathbf{r}'(1) = e\mathbf{i} + \mathbf{j} - \mathbf{k}$ and

$$\mathbf{T}(t) = (e^{2t} + 1 + t^{-4})^{-1/2}(e^t\mathbf{i} + \mathbf{j} - t^{-2}\mathbf{k})$$

Second, let us find $\mathbf{T}'(1)$. The straightforward expanding of the brackets leads to a long unpleasant computation, but instead of doing so we can use the Leibnitz rule $(f(t)\mathbf{v}(t))' = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t)$. Then we obtain that

$$\mathbf{T}'(t) = (e^{2t} + 1 + t^{-4})^{-1/2}(e^t\mathbf{i} + 2t^{-3}\mathbf{k}) + \frac{d(e^{2t} + 1 + t^{-4})^{-1/2}}{dt}(e^t\mathbf{i} + \mathbf{j} - t^{-2}\mathbf{k})$$

If $f(t)$ denotes $(e^{2t} + 1 + t^{-4})^{-1/2}$, then it follows that

$$\mathbf{T}'(1) = (e^2 + 2)^{-1/2}(e\mathbf{i} + 2\mathbf{k}) + f'(1)\mathbf{r}'(1)$$

One can compute that $f'(1) = -\frac{1}{2}(e^2 + 2)^{-3/2}(2e^2 - 4)$, but as we will see it has no impact on our computation. The vectors $\mathbf{r}'(1)$ and $\mathbf{T}'(1)$ are parallel to the tangent and the normal vectors, so we can use them to compute a normal vector \mathbf{x} to the osculating plane

$$\mathbf{x} = \mathbf{r}'(1) \times \left((e^2 + 2)^{-1/2}(e\mathbf{i} + 2\mathbf{k}) + f'(1)\mathbf{r}'(1) \right) = \mathbf{r}'(1) \times (e^2 + 2)^{-1/2}(e\mathbf{i} + 2\mathbf{k})$$

where the term with $f'(1)$ is cancelled because $\mathbf{r}'(1) \times \mathbf{r}'(1) = \mathbf{0}$. Finally, if we remove the scalar $(e^2 + 2)^{-1/2}$ from the expression, then we still get a normal vector to the osculating plane, hence

$$\mathbf{r}'(1) \times (e\mathbf{i} + 2\mathbf{k}) = (e\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (e\mathbf{i} + 2\mathbf{k}) = 2\mathbf{i} - 3e\mathbf{j} - e\mathbf{k}$$

is such a vector. Then, the plane is given by an equation $2x - 3ey - ez + d = 0$, and using that $\mathbf{r}(1) = (e, 1, 1)$ is in the plane, we get $d = 2e$.

10. (i) A typical difficulty was to realize that the wind can be represented by a vector $\mathbf{w} = (1, -1, 0)$ (regardless to its height, etc.)

(ii) Formally, the plane of the canopy is determined by points $P_1 = (220, 0, 75)$ and $P_2 = (0, 320, 50)$ and a parallel vector \mathbf{w} . Therefore, a normal vector can be found as $\overrightarrow{P_1P_2} \times \mathbf{w} = (220, -320, 25) \times (1, -1, 0)$, and the rest is easy (see the solution for the midterm for details).

(iii) A more laborious solution of this problem rests on the physical intuition that the plane of the canopy is fixed by P_1, P_2 and a third point $P_0 = (0, 0, z_0)$ of unknown height. So, a normal vector (depending on the unknown z_0) is $\mathbf{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_0P_1}$, and we can use that \mathbf{n} is perpendicular to the wind to find z_0 . Namely, we start with a vector equation $\mathbf{n} \cdot (\overrightarrow{P_1P_2} \times \overrightarrow{P_0P_1}) = 0$, expand it using the formulas and solve the obtained linear equation on z_0 . A certain simplification is obtained by observing that $\mathbf{n} \cdot (\overrightarrow{P_1P_2} \times \overrightarrow{P_0P_1})$ is the triple scalar product, so the equation on z_0 becomes

$$\begin{vmatrix} 1 & -1 & 0 \\ 220 & -320 & 25 \\ 220 & 0 & 75 - z_0 \end{vmatrix} = 0$$

It remains to expand it via the formula for the determinant and to solve the resulting linear equation on z_0 .

11. The problem is solved in two stages (and many points were lost in the second stage). The first part is standard: one gets that $x(t) = 28t \cos \theta$, hence $y = 20 \tan \theta - \frac{5}{2 \cos^2 \theta}$ when the ball hits the wall. The second stage is to maximize y as a function of θ by solving the equation $\frac{dy}{d\theta} = 0$.