# Math114-003; Review on multivariable calculus

Usually $w = F(x, y, z)$ will denote a function of three variables and $w = f(x, y)$ will denote a function of two variables. We prefer to work with $F(x, y, z)$ when possible, the case of $f(x, y)$ is similar.

**1. Graphs, level curves and level surfaces.**

- The graph of $f(x, y)$ is the set of triples $(x, y, w = f(x, y))$ where $(x, y)$ is a point in the domain of $f$.
- The level curves of $f(x, y)$ are given by $f(x, y) = c$.
- The level surfaces of $F(x, y, z)$ are given by $F(x, y, z) = c$.

Examples:

- The graph of a linear function $f(x, y) = ax + by + c$ is a plane; its level curves are lines.
- The level curves of $f(x, y) = x^2 - ay^2$ where $a > 0$ are hyperbolas for $c \neq 0$ and a cross (two intersecting lines) for $c = 0$. Saddle point at the origin.
- The level curves of $f(x, y) = x^2 + ay^2$ where $a > 0$ are ellipses for $c > 0$ and the origin point for $c = 0$. Local minimum at the origin.
- The level surfaces of $F(x, y, z) = x^2 + ay^2 + bz^2$ are ellipsoids for $c > 0$ (spheres when $a = b = 1$).

**2. Limits and continuity.**

- For $P_0$ in the domain of $F(x, y, z)$ the limit $\lim_{P \to P_0} F(P)$ is defined similarly to the case of a single variable, i.e. the value of $F(P)$ approaches the limit then $P$ approaches $P_0$ in the sense that $P \neq P_0$ and $|P - P_0| \to 0$ (we subtract the position vectors of $P$ and $P_0$, then $|P - P_0|$ is the distance between $P$ and $P_0$).
- Using the $\varepsilon$–$\delta$ terminology, one says that $\lim_{P \to P_0} F(P)$ exists and equals to $L$ if and only if for any positive $\varepsilon$ one can find a positive $\delta = \delta(\varepsilon)$ such that $|F(P) - F(P_0)| < \varepsilon$ for any $P$ with $P \neq P_0$ and $|P - P_0| < \delta$ (that is $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2$).
- Analogs of the usual limit laws hold, for example

$$\lim_{P \to P_0} (F(P) \cdot G(P)) = \lim_{P \to P_0} F(P) \cdot \lim_{P \to P_0} G(P)$$

- It is a little bit technical task to find a limit (or to prove that it does not exist) directly from the definition. Usually, it includes a play with estimates (inequalities) on $\varepsilon$ and $\delta$. Sometimes, one can use polar or spherical coordinates to compute limits, especially when $x^2 + y^2$ or $x^2 + y^2 + z^2$ appears in denominator. Often, a substitution can be used to reduce to the case of such a denominator.
- Example: to prove that $\lim_{(x,y,z) \to (0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0$ use spherical substitution $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$ ($\rho$ tends to 0, but $\phi$ and $\theta$ are arbitrary).
- Path criterion: there is no limit at $P_0$ if we can find different passes (e.g. lines through $P_0$) such that the limits of $f(P)$ along those

pathes are different as $P$ approaches $P_0$. For example, this criterion shows that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

- $F(x,y,z)$ is continuous at $P_0 = (x_0, y_0, z_0)$ if $F(P_0) = \lim_{P\to P_0} F(P)$.
- Similarly to the single variable case, a product, a sum, a composition, etc. of continuous functions is continuous.

**3. Partial derivatives.**

- For $P_0 = (x_0, y_0, z_0)$, the partial derivative $F_x(P_0) = \frac{\partial F}{\partial x}(P_0)$ is defined as $\frac{dF(x,y_0,z_0)}{dx}(x_0) = \lim_{x\to x_0} \frac{F(x,y_0,z_0)-F(P_0)}{x-x_0}$ (i.e. we derive $F$ with respect to $x$ while $y_0$ and $z_0$ are kept constant).
- Linearity: $(F+G)_x = F_x + G_x$ and $(CF)_x = C(F_x)$ for a constant $C$.
- Leibnitz rule: $(F \cdot G)_x = FG_x + F_xG$.
- If $z = f(x,y)$ is given implicitly by $F(x,y,z) = 0$ then one can find $\frac{\partial z}{\partial x}$ by applying $\frac{\partial}{\partial x}$ to the equation $F(x,y,z)$. So, we differentiate this equation with respect to $x$, where $y$ is fixed (treated as a constant) and $z$ is treated as a function of $x$. Using the chain rule (see below) we get $F_x + F_z\frac{\partial z}{\partial x} = 0$, so

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$

**4. Higher order partial derivatives.**

- Higher order partial derivatives are defined as $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial F}{\partial y})$, etc.
- Another notation: $F_{xy} = (F_x)_y$.
- Caution: $F_{yx} = \frac{\partial^2 F}{\partial x \partial y}$.
- A good new: by Clairaut's theorem, if all second order derivatives are continuous then the order is not important, namely $F_{xy} = F_{yx}$.
- Using Clairaut's theorem we can choose the order of differentiating which is more convenient computationally. For example, to compute $f_{xy}$ for $f(x,y) = x^x$ first differentiate with respect to $y$.

**5. Differentials and linearization.**

- If $F$ has continuous first order partial derivatives at $P_0 = (x_0, y_0, z_0)$, then it is differentiable at $P_0$, i.e. it admits a good linear approximation (or linearization) $F(x,y,z) \approx L(x,y,z)$ around $P_0$ for

$$L(x,y,z) = F(P_0) + F_x(P_0)\Delta x + F_y(P_0)\Delta y + F_z(P_0)\Delta z =$$
$$F(P_0) + F_x(P_0)(x-x_0) + F_y(P_0)(y-y_0) + F_z(P_0)(z-z_0)$$

- Symbolically, one introduces the full differential of $F(x,y,z)$ at $P_0$ as $dF = F_xdx + F_ydy + F_zdz$.
- If $L(x,y) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0)$ is the linearization of $w = f(x,y)$ at $P_0 = (x_0, y_0)$, then the equation $w = L(x,y)$ defines a plane which touches the graph of $f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$.
- The geometric meaning of $f(x,y)$ being differentiable at $P_0$ is that its graph admits a tangent plane at $(x_0, y_0, f(x_0, y_0))$.

**6. Chain rules.**

The chain rule gives a formula for partial derivatives of a composed function. There are many particular cases of the chain rule that can be produced using the tree diagram. Here we consider few examples. Let $w$ be a function of $x, y, z$, say $w = F(x, y, z)$.

- If $x, y$ and $z$ are functions of $t$, then $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$.
- If $u = h(w)$, then $\frac{\partial u}{\partial x} = \frac{du}{dw}\frac{\partial w}{\partial x}$, $\frac{\partial u}{\partial y} = \frac{du}{dw}\frac{\partial w}{\partial y}$ and $\frac{\partial u}{\partial z} = \frac{du}{dw}\frac{\partial w}{\partial z}$.
- If $x, y$ and $z$ are functions of $t$ and $s$, say $x = f(s, t)$, $y = g(s, t)$ and $z = h(s, t)$, then $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$ and $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$.
- Example: if $f(1, 2) = 3$, $g(1, 2) = 4$, $h(1, 2) = 5$, then $\frac{\partial w}{\partial s}(1, 2) = F_x(3, 4, 5)f_s(1, 2) + F_y(3, 4, 5)g_s(1, 2) + F_z(3, 4, 5)h_s(1, 2)$.

Formulate the analogs for $w = f(x, y)$ (actually, remove all terms involving $z$).

**7. Gradient, tangent planes, normal lines.**

Assume that $P_0 = (x_0, y_0, z_0)$ lies on a level surface $S$ given by $F(x, y, z) = C$, in other words $F(P_0) = C$.

- The gradient of $F(x, y, z)$ at $P_0$ is the vector $\nabla F(P_0) = F_x(P_0)\mathbf{i} + F_y(P_0)\mathbf{j} + F_z(P_0)\mathbf{k}$; it is normal to $S$ at $P_0$.
- $\nabla F(P_0)$ points in the direction of the fastest growth of $F(x, y, z)$ near $P_0$.
- The tangent plane to $S$ at $P_0$ is given by $F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$, so it is orthogonal to $\nabla F(P_0)$.
- The normal line to $S$ through $P_0$ is given parametrically by $x = x_0 + F_x(P_0)t, y = y_0 + F_y(P_0)t, z = z_0 + F_z(P_0)t$.

Formulate the analogs for $f(x, y)$ (level surfaces should be replaced with level curves).

**8. Directional derivatives.**

- If $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a unit vector, then the directional derivative in the direction of $\mathbf{u}$ is defined as
$$D_{\mathbf{u}}F(P_0) = \lim_{t \to 0} \frac{F(x_0 + at, y_0 + bt, z_0 + ct) - F(P_0)}{t}$$
- Directional derivative can be computed by the formula $D_{\mathbf{u}}F(P_0) = \nabla F(P_0) \cdot \mathbf{u}$.
- In particular, the direction of fastest increase (resp. decrease) is $\mathbf{v} = \frac{\nabla F(P_0)}{|\nabla F(P_0)|}$ (resp. $-\mathbf{v} = \frac{-\nabla F(P_0)}{|\nabla F(P_0)|}$). The directional derivative in that direction is $|\nabla F(P_0)|$ (resp. $-|\nabla F(P_0)|$ ).
- It also follows that the directional derivative $D_{\mathbf{u}}$ is zero if and only if $\mathbf{u}$ is perpendicular to the gradient, i.e. $\mathbf{u}$ lies in the tangent plane to the level surface at $P_0$.

Formulate the analogs for $f(x, y)$.

**9. Critical points.**

In this section we study only the case of $f(x, y)$. Assume that $P_0 = (x_0, y_0)$ is an inner point of a region $D$ in $\mathbf{R}^2$ (i.e. a small disc around $P_0$ is contained in $D$).

- $P_0$ is a critical point if either $f$ is not differentiable at $P_0$ or $\nabla f(P_0) = 0$.
- Any local extremum is a critical point.

Assume that $P_0$ is an inner point of $D$ and $f$ is twice differentiable at $P_0$.

- If $f_{xx}(P_0) > 0$ and $f_{xx}(P_0)f_{yy}(P_0) - f_{xy}(P_0)^2 > 0$ then $P_0$ is a local maximum.
- If $f_{xx}(P_0) < 0$ and $f_{xx}(P_0)f_{yy}(P_0) - f_{xy}(P_0)^2 > 0$ then $P_0$ is a local minimum.
- If $f_{xx}(P_0)f_{yy}(P_0) - f_{xy}(P_0)^2 < 0$ then $P_0$ is a saddle point.
- If $f_{xx}(P_0)f_{yy}(P_0) - f_{xy}(P_0)^2 = 0$ then no conclusion about $P_0$ can be made from this information.

Global extreme values can be found using the following facts.

- The boundary of a region $D$ is the set of points which are not an inner both for $D$ and for its complement in $\mathbf{R}^2$. For example, the unit circumference is the boundary of both open and closed unit discs.
- $D$ is closed if it contains its boundary $\delta(D)$. For example, closed discs are closed, but open discs are not closed.
- A continuous function $f(x, y)$ attains its extreme values on bounded closed regions.
- An absolute extreme value is attained either at a critical point or on the boundary.

**10. Lagrange multipliers.**

A local extremum point $P$ of a function $f(x, y, z)$ on a level surface $F(x, y, z) = k$ of $F$ is a solution of the following system of equations:

$$\begin{cases} F(P) = k \\ \nabla f(P) = \lambda \nabla F(P) \end{cases}$$

Here $P$ and $\lambda$ are unknowns. Since $\nabla F(P) = F_x(P)\mathbf{i} + F_y(P)\mathbf{j} + F_z(P)\mathbf{k}$ and $\nabla f(P) = f_x(P)\mathbf{i} + f_y(P)\mathbf{j} + f_z(P)\mathbf{k}$, the system can be rewritten as

$$\begin{cases} F(x_0, y_0, z_0) = k \\ f_x(x_0, y_0, z_0) = \lambda F_x(x_0, y_0, z_0) \\ f_y(x_0, y_0, z_0) = \lambda F_y(x_0, y_0, z_0) \\ f_z(x_0, y_0, z_0) = \lambda F_z(x_0, y_0, z_0) \end{cases}$$

Formulate the analogs for extreme points of $f(x, y)$ subject to a constraint $F(x, y) = k$.

To find extreme points of $f(x,y,z)$ subject to two constraints $g(x,y,z) = k$ and $h(x,y,z) = l$, solve the following system

$$\begin{cases} g(P) = k \\ h(P) = l \\ \nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P) \end{cases}$$

Here $P = (x,y,z)$ and $\lambda, \mu$ are five unknowns. The system consists of five equations because the vector equation on the gradients reduces to three separate equations on the coefficients of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$.

**11. Double integrals over rectangles.**

Let $R = [a,b] \times [c,d] = \{(x,y) | a \leq x \leq b, c \leq y \leq d\}$ be a rectangle and $f(x,y)$ be a continuous function on $R$. Divide $[a,b]$ to $m$ equal parts of size $\Delta x = (b-a)/m$ and divide $[c,d]$ to $n$ equal parts of size $\Delta y$, then we obtain a division of $R$ to $mn$ rectangles $R_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$. For any choice of points $(x_{ij}^*, y_{ij}^*)$ in each $R_{ij}$, we can form a double Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

where $\Delta A = \Delta x \Delta y$ is the area of any small rectangle. The double integral $\iint_R f(x,y) dA$ is defined as the limit of double Riemann sums when both $\Delta x$ and $\Delta y$ tend to zero. In particular, we can use the approximation

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A \approx \iint_R f(x,y) dA$$

both to approximate the double integral and to approximate the double sum (whichever is more difficult to compute directly).

**12. Iterated integrals and Fubini's theorem.**

Fubini's theorem is used to evaluate double integrals using iterated integrals.

Theorem: If $R = [a,b] \times [c,d]$ and $f(x,y)$ is continuous on $R$ (or, at least, $f(x,y)$ is bounded on $R$ and is discontinuous along finitely many smooth curves), then

$$\int_c^d \int_a^b f(x,y) dx dy = \iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx$$

The following easy fact often simplifies evaluation of iterated integrals: if $f(x,y)$ splits as $f(x,y) = g(x)h(y)$, then

$$\iint_R f(x,y) dA = \int_a^b g(x) dx \int_c^d h(y) dy$$

**13. Double integrals over general regions.**

If $D$ is any bounded region enclosed by finitely many smooth curves and $f(x,y)$ is continuous on $D$, then we define the double integral as

$$\iint_D f(x,y) dA = \iint_R F(x,y) dA$$

where $R$ is any rectangle containing $D$ and $F(x, y)$ equals to $f(x, y)$ on $D$ and vanishes outside of $D$ (in particular, $F(x, y)$ is continuous outside of the boundary of $D$).

This is a list of main properties of double integrals.

- Linearity: $\iint_D (f(x,y)+g(x,y))dA = \iint_D f(x,y)dA + \iint_D g(x,y)dA$, and $\iint_D Cf(x,y)dA = C\iint_D f(x,y)dA$ for a constant $C$.
- Connection to areas: $\iint_D dA = A(D)$ is the area of $D$.
- Connection to volumes: if $f(x, y)$ is non-negative on $D$, then the volume below its graph over $D$ is $\iint_D f(x,y)dA$.
- More generally, if $f(x, y) \leq g(x, y)$ on $D$ then the volume of the solid over $D$ enclosed by the graphs of $f(x, y)$ and $g(x, y)$ equals to $\iint_D (g(x,y) - f(x,y))dA$.
- Average value: the average of $f(x, y)$ on $D$ is $f_{ave} = A(D)^{-1}\iint_D f(x,y)dA$ (so the average of a constant $C$ is $A(D)^{-1} \cdot CA(D) = C$).
- If $f(x, y) \leq g(x, y)$ on $D$, then $\iint_D f(x,y)dA \leq \iint_D g(x,y)dA$
- If $D$ is the union of two regions $D_1, D_2$ which do not overlap except perhaps on their boundaries, then $\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$.

**14. Double integrals over type I and type II regions.**

- If $g_1$ and $g_2$ are continuous and $g_1(x) \leq g_2(x)$ on $[a; b]$, then the region given by $a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$ is called a type I region.
- A type I region can be sliced nicely by vertical lines. The length of such a slice by the line $x = x_0$ equals to $g_2(x_0) - g_1(x_0)$.
- Similarly, if $h_1$ and $h_2$ are continuous and $h_1(y) \leq h_2(y)$ on $[c; d]$, then the region given by $c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$ is called a type II region.
- A type II region can be sliced nicely by horizontal lines. The length of such a slice by the line $y = y_0$ equals to $h_2(y_0) - h_1(y_0)$.
- Usually type I region (or regions) shows up when the boundary of the region is given by the curves of the form $y = f(x)$, and type II region (or regions) shows up when the boundary of the region is given by the curves of the form $x = g(y)$.

Fubini's theorem implies that double integrals over type I and type II regions can be evaluated by a special formula.

- Fubini's theorem for a type I region:

$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

- Notice that we first integrate on $y$ along vertical slices of $D$ (with fixed $x$).

- Fubini's theorem for a type II region:

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

- Notice that we first integrate on $x$ along horizontal slices of $D$ (with fixed $y$).

If $D$ is both type I and type II region, then we can use Fubini's theorem to switch the order of integration in an iterated integral. This trick is often used to evaluate repeated integrals.

**15. Integration in polar coordinates.**

- Sometimes, it is convenient to switch to polar coordinates to simplify the integrand or the region of integration (or both).
- A polar rectangle $R$ is given by conditions $\alpha \le \theta \le \beta, a \le r \le b$. It is the region enclosed by circumferences of radii $a$ and $b$ and two rays.
- For example: a circle of radius $R$ is a polar rectangle with $0 \le \theta \le 2\pi, 0 \le r \le R$. A part of the unit circle cut off by $x \le y$ is a polar rectangle given by $\frac{5\pi}{4} \le \theta \le \frac{9\pi}{4}, 0 \le r \le 1$ (or, that is equivalent, $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}, 0 \le r \le 1$).
- To evaluate a double integral as an iterated integral in polar coordinates use the symbolic formula $dA = rdrd\theta$. In particular, for $R$ as above one has

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

- More generally, $D$ is a type I polar region if it is given by the conditions $\alpha \le \theta \le \beta, g_1(\theta) \le r \le g_2(\theta)$. For such $D$ one has

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta$$

- Caution: do not forget the factor of $r$.

**16. Surface area.**

- If $f(x,y)$ is a function over a region $D$ with continuous partial derivatives, then the graph $z = f(x,y)$ of $f$ over $D$ is a smooth surface $S$, and its area can be found as $A(S) = \iint_D \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}dA$

**17. Triple integrals.**

Triple integral of a function $F(x,y,z)$ over a solid region $E$ in $\mathbf{R}^3$ is defined similarly to the double integral. If $B = [a,b] \times [c,d] \times [r,s]$ is a rectangular box, then we use tripe Riemann sums over small boxes with edges $\Delta x, \Delta y, \Delta z$ (their volume is $\Delta V = \Delta x \Delta y \Delta z$). For any bounded solid region $E$ we extend $F$ by zero outside of $E$ and integrate the new function over a large box $B$. Triple integrals behave very similarly to double integrals.

- Linearity:

$$\iiint_E (f(x,y,z)+g(x,y,z))dV = \iiint_E f(x,y,z)dV + \iiint_E g(x,y,z)dV$$

  and $\iiint_E Cf(x,y,z)dV = C\iiint_E f(x,y,z)dV$ for a constant $C$.
- Connection to volumes: $\iiint_E dV = V(E)$ is the volume of $E$.
- Average value: the average of $f(x,y,z)$ on $E$ is $f_{ave} = V(E)^{-1}\iiint_E f(x,y,z)dV$ (so the average of a constant $C$ is $V(E)^{-1}\cdot CV(E) = C$).
- If $f(x,y,z) \le g(x,y,z)$ on $E$, then $\iiint_E f(x,y,z)dV \le \iiint_E g(x,y,z)dV$
- If $E$ is the union of two regions $E_1, E_2$ which do not overlap except perhaps on their boundaries, then $\iiint_E f(x,y,z)dV = \iiint_{E_1} f(x,y,z)dV + \iiint_{E_2} f(x,y,z)dV$.

**18. Fubini's theorem and iterated triple integrals.**

We give a list of consequences of Fubini's theorem for various regions. Reduction to double integrals.

- For a parallelepiped $B = [a,b]\times[c,d]\times[r,s]$ we have $\iiint_B f(x,y,z)dV = \int_a^b\int_c^d\int_r^s f(x,y,z)dzdydx$, and similarly for other orders of integration (there are six different orders).
- We say that $E$ is a type 1 region if it is of the form $\{(x,y,z)|(x,y)\in D, u_1(x,y)\le z\le u_2(x,y))\}$ for a region $D$ in $\mathbf{R}^2$ and two functions $u_1(x,y)\le u_2(x,y)$ on $D$. For such $E$ one has

$$\iiint_E f(x,y,z)dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdA$$

- In particular,

$$V(E) = \iiint_E dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} dzdA = \iint_D (u_2(x,y)-u_1(x,y))dA$$

  as we already know from the properties of double integrals.
- If $f$ splits as $g(x,y)h(z)$ and $E$ splits as $E = D\times[r,s]$ (i.e. $E$ is a cylinder over $D$), then the integral splits as

$$\iiint_E f(x,y,z)dV = \iint_D g(x,y)dA\int_r^s h(z)dz$$

- We say that $E$ is a type 2 (resp. 3) region if it is of the form $\{(x,y,z)|(y,z)\in D, u_1(y,z)\le x\le u_2(y,z))\}$ (resp. $\{(x,y,z)|(x,z)\in D, u_1(x,z)\le y\le u_2(x,z))\}$) for a region $D$ in the $yz$ plane (resp. $xz$ plane) and two functions $u_1\le u_2$ on $D$.
- Formulate the similar formulas for type 2 and 3 regions.

Reduction to iterated triple integrals.

- If $E$ is a type 1 region over a plain region $D$ which in its turn is a type I region, then $E$ is given by inequalities $a\le x\le b, g_1(x)\le y\le g_2(x), u_1(x,y)\le z\le u_2(x,y)$.

- We can use Fubini's theorem for $D$ to express double integrals over $D$ as an iterated double integral, so if $E$ is a type 1 solid region over a type I plain region $D$, then we obtain the following formula for the triple integral

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx$$

- Write down five similar formulas for other orders of integration accordingly to the cases of a type 1 region over a type II region, a type 2 region over a type I region, etc.

**19. Triple integrals in cylindrical and spherical coordinates.**

- To change to cylindrical coordinates use the symbolic formula

$$dV = rdzdrd\theta$$

and adjust the limits of integration by describing a Cartesian solid $E$ in cylindrical coordinates.

- For example, if $E$ is a cylinder given by $x^2 + y^2 \leq R^2$, $r \leq z \leq s$, then

$$\iiint_E f(x,y,z)dV = \int_0^{2\pi} \int_0^R \int_r^s f(r\cos\theta, r\sin\theta, z)rdzdrd\theta$$

- To change to spherical coordinates use the symbolic formula

$$dV = \rho^2 \sin\phi d\rho d\theta d\phi$$

and adjust the limits of integration by describing a Cartesian solid $E$ in cylindrical coordinates.

- For example, if $E$ is a sphere given by $x^2 + y^2 + z^2 \leq R^2$, then

$$\iiint_E f(x,y,z)dV = \int_0^{\pi} \int_0^{2\pi} \int_0^R f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi$$

**20. Multiple integrals and change of variables.**

Assume that $T(u,v) = (x(u,v), y(u,v))$ gives a one-to-one map of a region $D'$ in the $uv$-plane onto a region $D$ in the $xy$-plane. We would like to express an integral over $D$ in the $xy$-plane using the new $u, v$ coordinates For example, for polar change of coordinates one takes $u = r, v = \theta$, and then $x = r\cos\theta, y = r\sin\theta$.

Define the Jacobian of $T$ as a $2 \times 2$ determinant of partial derivatives

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The area differential in new coordinates can be written as

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA'$$

where $dA$ is the area differential in the $xy$-plane and $dA'$ is the area differential in the $uv$-plane. Therefore we obtain the following change of variables formula:

$$\iint_D f(x,y)dA = \iint_{D'} f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA'$$

One should remember to take the absolute value of the Jacobian in this formula.

The three-dimensional analog for a transformation

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$$

is very similar but with $3\times 3$ Jacobian determinant used to relate the volume differentials in $xyz$ and $uvw$ spaces.

$$dV = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV'$$

Exercise: check that the formulas for polar, cylindrical and spherical integration are particular cases of the general change of variables formula. For example, check that

$$\left| \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} \right| = \rho^2 \sin\phi$$