Mathematics 114-003, Fall 2007

We give an example of solving a linear second-order ode with power series. The solution looks more difficult than it actually is because all small stages are explicitly written down. Usually, some of them should be done mentally. Also, stages (5) and (6) are so standard that you may wish to memorize them.

Problem. Solve the differential equation $y'' + (1+x)y' - y = 0$ by a power series expansion about $x = 0$.

Assume that $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution, then

$$y' = \left(\sum_{n=0}^{\infty} c_n x^n\right)' = \sum_{n=0}^{\infty} (c_n x^n)' = \sum_{n=0}^{\infty} n c_n x^{n-1} \tag{1}$$

Multiplying by $1+x$ we obtain that

$$(1+x)y' = (1+x)\sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1} + x\sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=1}^{\infty} n c_n x^n \tag{2}$$

Differentiating (1) we obtain that

$$y'' = \left(\sum_{n=1}^{\infty} n c_n x^{n-1}\right)' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \tag{3}$$

Summarizing the results, we see that

$$0 = y'' + (1+x)y' - y = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n \tag{4}$$

(A small trick we used here is that $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$.) Now, the key step is to shift the indexes as follows. Set $k = n-1$, then $n = k+1$ and we obtain from (1) that

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k \tag{5}$$

Choosing $k = n-2$, we similarly obtain from (3) that

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k \tag{6}$$

So, equation (4) can be rewritten as

$$0 = \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k - \sum_{k=0}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

Since the coefficient of x_k in the power expansion of 0 must vanish, we obtain that

$$(k+2)(k+1)c_{k+2} + (k+1)c_{k+1} + kc_k - c_k = 0$$

So, we obtain the following recursion on c_k's

$$c_{k+2} = -\frac{(k+1)c_{k+1} + (k-1)c_k}{(k+2)(k+1)}$$

For example, the initial value problem $y(0) = 1, y'(0) = -1$ is solved as follows: $c_0 = y(0) = 1$, $c_1 = y'(0) = -1$, for $k = 0$ we get $c_2 = -\frac{c_1 - c_0}{1 \cdot 2} = 1$, for $k = 1$ we get $c_3 = -\frac{2c_2 + 0c_1}{2 \cdot 3} = -\frac{1}{3}$. Thus, $y = 1 - x + x^2 - x^3/3 + \dots$