Math114-003; Review on vectors and vector functions

1. Three-dimensional space

- Cartesian coordinate axes in the space are three perpendicular axis: x-, y- and z-axis.
- A *Cartesian coordinate system* in the three-dimensional space describes each point P as a triple (x, y, z), where x is the coordinate of the projection of P on the x-axis, etc.
- Thus, a Cartesian coordinate system identifies the space with the set $\mathbf{R}^3$ of ordered triples of real numbers.
- The Pythagoras theorem implies that for $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$, the distance between these points can be found as $|P_0P_1| = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}$.
- In particular, the equation $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$ defines a sphere of radius R and with center at (x_0, y_0, z_0).
- In general (but not always!) an equation $F(x, y, z) = 0$ defines a surface and a pair of equations $F(x, y, z) = G(x, y, z) = 0$ defines a curve.

2. Vectors.

- A *vector* is an oriented interval from A to B, and it is denoted $\overrightarrow{AB}$. Two vector are equal if they have equal length and directions.
- We will use bold letters to denote vectors, e.g. $\mathbf{u}$ and $\mathbf{v}$. The length of $\mathbf{u}$ is denoted $|\mathbf{u}|$. The only vector of zero length is denoted $\mathbf{0}$.
- Any vector can be realized as $\mathbf{a} = \overrightarrow{OA}$ for some choice of A (O is the origin), and conversely, to any A we can attach the vector $\overrightarrow{OA}$ called the position vector of A. If $A = (a_1, a_2, a_3)$, then we say that a_1, a_2, a_3 are the *components* of $\mathbf{a}$ and write $\mathbf{a} = (a_1, a_2, a_3)$.
- More generally, if $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, then $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$.
- The vectors $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ are the unit vectors in the directions of the coordinate axes.

3. Addition and multiplication by a scalar.

- The addition is defined as $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (to find $\mathbf{u} + \mathbf{v}$ translate $\mathbf{v}$ so that its initial point is the terminal point of $\mathbf{u}$).
- For a scalar α, we define $\alpha\mathbf{u}$ so that $|\alpha\mathbf{u}| = |\alpha||\mathbf{u}|$, and $\alpha\mathbf{u}$ and $\mathbf{u}$ have the same direction (resp. opposite direction) if $\alpha > 0$ (resp. $\alpha < 0$). Also, $0\mathbf{u} = \mathbf{0}$.
- This operations can be done *componentwise*, i.e. $\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$ and $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.
- In particular, one can work with addition and multiplication by a scalar similarly to the usual addition and multiplication, e.g. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$, $\mathbf{a} + (-1)\mathbf{a} = \mathbf{0}$, etc.

- Note that $(a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. The triple $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is called the standard basis of $\mathbf{R}^3$, because any vector admits a unique representation as a linear combination of these vectors.

4. Dot product.

- The *dot (or scalar) product* in $\mathbf{R}^3$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. The definition in $\mathbf{R}^2$ is similar.
- The main geometric property of the dot product is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\alpha)$$

  where α is the angle between the vectors (from 0 to π).
- In particular, $\mathbf{a} \cdot \mathbf{a} = |a|^2$, and $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. Note that $\mathbf{0}$ is perpendicular to everything.
- Other basic formulas are: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$,$(\alpha\mathbf{a}) \cdot \mathbf{b} = \alpha(\mathbf{a} \cdot \mathbf{b})$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
- The angles α, β and γ of $\mathbf{a}$ with the coordinate axes are called *directional angles* of $\mathbf{a}$. Using the dot product with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ one finds that $\cos(\alpha) = |\mathbf{a}|^{-1}(\mathbf{a} \cdot \mathbf{i})$ and similarly for other angles. In particular, the unit vector in the direction of $\mathbf{a}$ can be expressed as $|\mathbf{a}|^{-1}\mathbf{a} = (\cos(\alpha), \cos(\beta), \cos(\gamma))$.
- The length of the projection of $\mathbf{b}$ onto $\mathbf{a}$ and the projection itself are given as

$$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \qquad \mathrm{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

5. Determinants.

A 2×2 *determinant* is a number assigned to 2 pairs of numbers (a_1, a_2) and (b_1, b_2)

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

The geometric meaning of the 2×2 determinant D is that its absolute value equals to the area of the parallelogram determined by $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, i.e. $|D| = |\mathbf{a}||\mathbf{b}| \sin(\alpha)$. In particular, it vanishes if an only if the vectors are parallel.

A 3×3 *determinant* is a number assigned to 3 triples of numbers (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3)

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

The geometric meaning of the 3×3 determinant D is that its absolute value equals to the volume of the parallelepiped determined by $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$. In particular, it vanishes if an only if the vectors are *coplanar* (i.e. lie in the same plane).

Remarks: in linear algebra, determinant is viewed as a function of a square array of numbers; the sign of the determinant can be found from the orientation of the vectors using the right hand rule.

6. Cross product.

- The *cross (or vector) product* exists only in $\mathbf{R}^3$. It is defined as $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$.
- An equivalent formula which can be memorized is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

- The main geometric property of the cross product is that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and its direction can be found from the right hand rule; its length is the area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$, so

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\alpha)$$

  where α is the angle between the vectors (from 0 to π).
- In particular, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel. Note that $\mathbf{0}$ is parallel to everything.
- Other basic formulas are: $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$,$(\alpha\mathbf{a})\times\mathbf{b} = \alpha(\mathbf{a}\times\mathbf{b})$ and $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- Caution: the latter formula indicates that we should work very carefully with the cross product.
- An alternative way to compute $\mathbf{a} \times \mathbf{b}$ is to expand the brackets in the expression $(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$ and to use that $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$; $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$; and $\mathbf{j}\times\mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

Also, there are more exotic formulas regarding triple products.

- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
- The latter triple product is called scalar triple product, and it equals to the volume of the parallelepiped determined by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. In particular, it vanishes if an only if the vectors are coplanar (i.e. lie in the same plane) and

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

7. Lines and planes.

Lines:

- Usually, a line is given by a point $P_0 = (x_0, y_0, z_0)$ and a direction vector $\mathbf{v} = (a, b, c)$ (clearly, there are many ways to give a line in such a way).

- A *vector equation* of this line is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ (here $\mathbf{r}$ is a function of a parameter t and $\mathbf{r}_0 = (x_0, y_0, z_0)$ is the position vector of P_0).
- The latter equation can be expanded as $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$. These are called *parametric equations*.
- Solving for t, one gets *symmetric equations*: $(x-x_0)/a = (y-y_0)/b = (z-z_0)/c$ (for non-zero a, b and c).
- A line segment between $\mathbf{r}_0$ and $\mathbf{r}_1$ is described as $\mathbf{r} = t\mathbf{r}_0 + (1-t)\mathbf{r}_1$ for $0 \le t \le 1$.

Planes:

- A plane can be given by a point $P_0 = (x_0, y_0, z_0)$ and a *normal vector* $\mathbf{n} = (a, b, c)$ (i.e. a perpendicular vector). Clearly, there are many ways to give a line in such way.
- A *vector equation* of this plane is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ (it is an equation on $\mathbf{r}$; $\mathbf{n}$ and $\mathbf{r}_0$ are given constant vectors).
- This equation can be expanded as $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$, and taking $d = -(ax_0 + by_0 + cz_0)$ we obtain a *linear equation* of the plane $ax + by + cz + d = 0$. The latter is unique up to multiplying the quadruple (a, b, c, d) by a non-zero number.
- The distance between a point $P_1 = (x_1, y_1, z_1)$ and a plane $ax + by + cz + d = 0$ equals to

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

There are many various questions about lines and planes, but we consider just few examples.

- To find the angle between two lines choose two vectors in their direction and compute the angle between them using the dot product.
- To find the angle between two planes find the angle between normal vectors. A normal vector to $ax + by + cz + d = 0$ is (a, b, c).
- To find a plane through three points P_0, P_1, P_2 with position vectors $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2$ find its normal vector as $\mathbf{n} = (\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0)$ (we use that $\mathbf{n}$ is perpendicular to the vectors $\mathbf{r}_1 - \mathbf{r}_0$ and $\mathbf{r}_2 - \mathbf{r}_0$ that are parallel to the plane). It gives you (a, b, c), and in order to find d use that a point P_0 is in the plane (you can use P_1 or P_2 as well).
- To find the intersection line of two planes, find any intersection point by solving the system of two linear equations, and use that this line is parallel to the cross product of the normal vectors (because it is perpendicular to both normal vectors).

8. Cylindrical and spherical coordinates.

Cylindrical coordinates.

- A point is described by a triple (r, θ, z), where z is the Cartesian z-coordinate and (r, θ) are the polar coordinates of the projection on the xy-plane.

- Remark: all values of r and θ are allowed, so any point possesses many possible representations in these coordinates. A unique choice can be ensured by an extra-condition $r \geq 0$, $0 \leq \theta < 2\pi$.
- $r = C$ defines a cylinder parallel to the z-axis and of radius C .
- Cylindrical to Cartesian: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.
- Cartesian to cylindrical: $r^2 = x^2 + y^2$, $\tan\theta = y/x$, $z = z$.

Spherical coordinates.

- A point P is described by a triple (ρ, θ, ϕ), where θ is the cylindrical θ-coordinate, ρ is the distance from the origin (i.e. $\rho = |OP|$) and ϕ is the angle between OP and the z-axis.
- Remark: one can restrict to the case when $\rho \geq 0$ and $0 \leq \phi \leq \pi$ (as in the text), but all formulas work for all values of the coordinates.
- $\rho = C$ defines a sphere with center at the origin and of radius C.
- Spherical to cylindrical: $r = \rho\sin\phi$, $\theta = \theta$, $z = \rho\cos\phi$.
- Cylindrical to spherical: $\rho^2 = r^2 + z^2$, $\theta = \theta$, $\tan\phi = r/z$.
- Spherical to Cartesian: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.
- Cartesian to spherical: $\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = y/x$, $(\tan\phi)^2 = (x^2 + y^2)/z^2$.
- Remark: one can easily deduce some of these formulas from the others. No need to memorize everything!

Example: the surface $\rho^2(\sin^2\phi\cos^2\theta + \cos^2\phi) = 4$ can be described as $x^2 + z^2 = 4$ in the Cartesian coordinates, so we get a cylinder along the y-axis of radius 2.

9. Vector functions.

- A *vector function* is a function from $\mathbf{R}$ to $\mathbf{R}^2$ or $\mathbf{R}^3$, e.g. $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, a \leq t \leq b$.
- We denote vector functions by bold letters, to distinguish them from *scalar functions* from $\mathbf{R}$ to itself.
- One can add, subtract, compute limits, check continuity, differentiate and integrate vector functions componentwise, for example, $(f(t), g(t))' = (f'(t), g'(t))$ and

$$\int_a^b (f(t), g(t), h(t))\, dt = \left(\int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt\right)$$

- The fundamental theorem of calculus for vector functions is

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^t \mathbf{r}(s)ds$$

- The base change rule is $\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds}$ for a scalar function $t = \phi(s)$.
- There are three Leibnitz rules regarding multiplication by a scalar function, and dot and cross products:

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$$

10. Parametric curves, space curves and tangent vectors.

- A vector function $\mathbf{r}(t)$ defines a *parametric curve* C.
- If we forget about the parameterization, then we get a *space (or plane)* curve which is the set of all values of $\mathbf{r}$ (or the range of $\mathbf{r}$). So, the parametric curve can be viewed as a space curve traced by a particle at some (depending on t) speed.
- There are many ways to parameterize the same space curve. Actually, $\mathbf{r}(\phi(s))$ is another parameterization of $\mathbf{r}(t)$ for any scalar function $t = \phi(s)$ (the substitution just changes our speed on the curve).
- A typical example: both $(\cos t, \sin t)$ and $(\cos t^3, \sin t^3)$ define the same plane curve (unit circle).
- Parametric curve C given by $\mathbf{r}(t)$ is *smooth* if f, g, h have continuous derivatives which do not vanish simultaneously.
- Caution: it may happen that the space curve is smooth at t_0 even when $\mathbf{r}'(t_0) = 0$. For example, $(\cos t^3, \sin t^3)$ is smooth at $t = 0$, but $\mathbf{r}'(0) = \mathbf{0}$. On the other hand, the plane curve (t^2, t^3) is not smooth at $t = 0$.
- The derivative $\mathbf{r}'(t_0)$ is a *tangent vector* to C at $\mathbf{r}(t_0)$ (if it is not zero!).
- The *unit tangent vector* is defined as the normalized tangent vector, i.e. $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

11. Arc length, normal vector and the osculating plane.

- The length of a smooth curve $\mathbf{r}(t)$, $a \le t \le b$ is defined as

$$L = \int_a^b |\mathbf{r}'(t)|dt = \int_a^b \sqrt{(f'(t)^2) + (g'(t)^2) + (h'(t)^2)}dt$$

- The arc length function is defined as the length of the part of the curve from an initial point $\mathbf{r}(t_0)$ to $\mathbf{r}(t)$. Thus $s(t) = \int_{t_0}^t |\mathbf{r}'(x)|dx$ and therefore $\frac{ds}{dt} = |\mathbf{r}'(t)|$ is the length of the tangent vector.
- If we invert $s(t)$ as $t = t(s)$ (that can be computationally difficult), then we can parameterize C by the arc length as $\mathbf{r}(t(s))$.
- The parameterization by arc length is unique, because it depends only on the space curve (and its orientation). So, depolymerization by an arc length is a good way to study properties of the space curve. Many invariants of the space curve (e.g. curvature) are defined using arc length parameterization.
- Usually, it is difficult to invert $s(t)$, but we can at least find derivative of $t(s)$ by the theorem on derivatives of inverse functions, so $\frac{dt}{ds} = \left(\frac{ds}{dt}\right)^{-1} = |\mathbf{r}'(t)|^{-1}$.

- Then it follows from the chain rule that $|\frac{d\mathbf{r}}{ds}| = 1$. Hence, $\frac{d\mathbf{r}}{ds}$ is the unit tangent vector to the curve, and parameterization with the arc length means that we travel on our curve with the constant speed 1.
- The curvature is defined (theoretically) as $\kappa = |\frac{d\mathbf{T}}{ds}|$. Intuitively, it is the rate of change of the direction of the tangent vector. Two practical formulas to compute it are as follows

$$\frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

  In both formulas the differentiation is with respect to t, so there is no need to respirometric the curve!
- For a plane curve $y = f(x)$, the latter formula gives

$$\kappa = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

- The vector $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$, and its normalization $\mathbf{N}(t)$ is called the *normal vector* to C. So,

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

- The *binormal vector* $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$.
- The plane through $\mathbf{r}(t)$ and parallel to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ is called the osculating plane. This is the closest plane to C through $\mathbf{r}(t)$. If C lies in a plane M, then the osculating plane at any point is just M.
- To find a linear equation $ax + by + cz + d = 0$ of the osculating plane, you can take $(a, b, c) = \mathbf{B}(t)$ (because $\mathbf{B}(t)$ is normal to the osculating plane) and then find d from the fact that $\mathbf{r}(t)$ lies in the plane.

12. Velocity and acceleration.

- Assume that a particle moves in the space, and $\mathbf{r}(t), \mathbf{v}(t), \mathbf{a}(t)$ and $v(t)$ are its position, velocity, acceleration and speed, respectively. Then $\mathbf{r}' = \mathbf{v}$, $\mathbf{r}'' = \mathbf{v}' = \mathbf{a}$ and $|\mathbf{v}| = v$ at any time t.
- In particular, $\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(x)dx$ (notice that $\mathbf{r}(t) - \mathbf{r}(t_0)$ is the displacement) and $\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(x)dx$.
- For example, if the only external force is due to gravity, then the acceleration is constant: $\mathbf{a}(t) = (0, -g)$. So, if $\mathbf{v}(0) = (v_x(0), v_y(0))$ is the initial velocity and $\mathbf{r}(0) = (x_0, y_0)$ is the initial position, then

$$\mathbf{v}(t) = (v_x(0), v_y(0) - gt)$$

$$(x(t), y(t)) = \left(x_0 + v_x(0)t, y_0 + v_y(0)t - \frac{gt^2}{2}\right)$$

- The acceleration vector always lies in the osculating plane, hence $\mathbf{a}(t)$ is a linear combination of $\mathbf{T}(t)$ and $\mathbf{N}(t)$, i.e.
$$\mathbf{a}(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t)$$
for some scalar functions $a_T(t)$ and $a_N(t)$ (no contribution of $\mathbf{B}(t)$ in the expression for $\mathbf{a}(t)$).
- One calls a_T and a_N the *tangential* and the *normal components* of the acceleration.
- The tangential component is just the change of rate of the speed. To compute it you can use either of the following two formulas
$$\frac{dv}{dt} = a_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|}$$
- The normal component can be computed from either of the following two formulas
$$\kappa v^2 = a_N = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}$$
In particular, $a_N \geq 0$.
- Using these formulas, we can represent the acceleration as
$$\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}$$