Solutions to Homework 4

February 10, 2007

8.9

#16. The characteristic polynomial of $\mathbf{A}$ turns out to be $-\lambda^3+2\lambda^2+\lambda-2$. Now, by Cayley-Hamilton Theorem, we know that

$$\begin{aligned} & -\mathbf{A}^3 + 2\mathbf{A}^2 + \mathbf{A} - 2\mathbf{I} = 0 \\ \Rightarrow \quad & 2\mathbf{I} = -\mathbf{A}^3 + 2\mathbf{A}^2 + \mathbf{A} \\ \Rightarrow \quad & 2\mathbf{A}^{-1} = -\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I} \qquad \text{(Multiplying both sides by } \mathbf{A}^{-1}\text{)} \\ \Rightarrow \quad & \mathbf{A}^{-1} = \tfrac{-1}{2}\mathbf{A}^2 + \mathbf{A} + \tfrac{1}{2}\mathbf{I}. \end{aligned}$$

Thus, once we calculate $\mathbf{A}^2$, we can find $\mathbf{A}^{-1}$. It turns out that

$$\begin{pmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \end{pmatrix}.$$

8.10

#2. The respective eigenvalues corresponding to the given eigenvectors are 2, 2 and -1.

#8. No, it is not orthogonal because, the matrix is not invertible.

#9. No, it is not orthogonal, because the columns are not unit vectors i.e. do not have magnitude 1.

#21. The eigenvalues corresponding to the eigenvectors $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$ are -2, -2 and 4 respectively. We notice that both $\mathbf{K}_1$ and $\mathbf{K}_2$ are orthogonal to $\mathbf{K}_3$ whereas $\mathbf{K}_1$ and $\mathbf{K}_2$ are not orthogonal to each other. So, we shall use Gram-Schmidt process to orthogonalise $\mathbf{K}_1$ and $\mathbf{K}_2$. So, our new set of orthogonal vectors $\{\mathbf{V}_1, \mathbf{V}_2\}$ are defined as follows:

$$\begin{aligned}
\mathbf{V}_1 &= \mathbf{K}_1 \\
\mathbf{V}_2 &= \mathbf{K}_2 - \tfrac{\mathbf{K}_2\cdot\mathbf{V}_1}{\mathbf{V}_1\cdot\mathbf{V}_1}\mathbf{V}_1 \\
&= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix}.
\end{aligned}$$

But, now we have to normalise these vectors i.e. make them unit length by dividing them by their magnitude. So, in our case the orthogonal matrix $\mathbf{P}$ will be

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

#25. To show $\mathbf{AB}$ is orthogonal, it is enough to show that $(\mathbf{AB})^T = (\mathbf{AB})^{-1}$. Since, $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, we know that $\mathbf{A}^T = \mathbf{A}^{-1}$ and $\mathbf{B}^T = \mathbf{B}^{-1}$.

$$\begin{aligned}
(\mathbf{AB})^T &= \mathbf{B}^T\mathbf{A}^T \\
&= \mathbf{B}^{-1}\mathbf{A}^{-1} \quad \text{(Because } \mathbf{A} \text{ and } \mathbf{B} \text{ are orthogonal)} \\
&= (\mathbf{AB})^{-1}.
\end{aligned}$$

Hence, $\mathbf{AB}$ is orthogonal.

8.12

#2. Yes, it is diagonalisable. The diagonalising matrix and the diagonalised matrix are

$$\mathbf{P} = \begin{pmatrix} -5 & -1 \\ 4 & 2 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}.$$

#13. Yes, it is diagonalisable. It has distinct eigenvalues viz. 0, 1 and 2 and the diagonalising matrix and the diagonalised matrix are

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

#18. No, it is not diagonalisable. We have only one eigenvalue viz. 1 and corresponding to that eigenvalue, we have only one linearly independent eigenvector

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Hence, the matrix is not diagonalisable.

#24. The eigenvalues are -1 and 3 and the corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{K}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{respectively.}$$

The orthogonal diagonalising matrix and the diagonalised matrix are

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

#26. The eigenvalues are -1, -1 and 5 and the corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{K}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{respectively.}$$

However, we have to orthogonalise $\mathbf{K}_1$ and $\mathbf{K}_2$ and we use Gram-Schmidt process. It turns out that it is enough to replace $\mathbf{K}_2$ by $\mathbf{V}_2$ where

$$\mathbf{V}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}.$$

So, our orthogonal diagonalising matrix and the diagonalised matrix are

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$