

Math 114-004, Fall 2009

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Calculus of Vector Functions

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Maple Demonstration

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The **unit tangent vector** is

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

To compute the derivative, if f, g, h are differentiable,

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

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So $\vec{r}'(0) = 3\vec{j} + 4\vec{k}$. $|\vec{r}'(0)| = \sqrt{9 + 16} = 5$. Hence

$$\vec{T} = \frac{\vec{r}'(0)}{|\vec{r}'(0)|} = \frac{3}{5}\vec{j} + \frac{4}{5}\vec{k}$$

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$$x = 1 + 5t, \quad y = 1 + 4t \quad z = 1 + 3t$$

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By $\vec{r}(0) = \vec{0}$ and $\vec{r}(0) = -\vec{i} + \vec{C}$, $\vec{C} = \vec{i}$. Hence,

$$\vec{r}(t) = (-\cos(t) + 1)\vec{i} - \sin(t)\vec{j} + t^2\vec{k}$$