Math 114-004, Fall 2009

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Equations of Lines and Planes in 3D

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$$\Uparrow$$

Vectors and Vector Algebra

Equations of Lines in 3D

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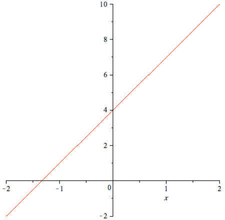
Recall: What do we need to define the equation of a line in 2D?



Equations of Lines in 3D

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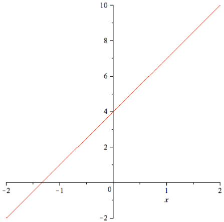

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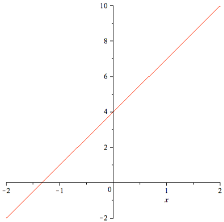

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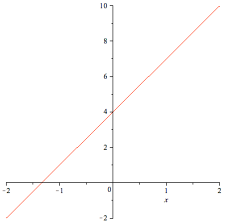

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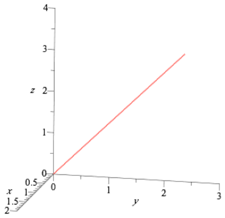
$$y - y_0 = s(x - x_0) \Rightarrow y = sx - sx_0 + y_0$$

What do we need to define the equation of a line in 3D?



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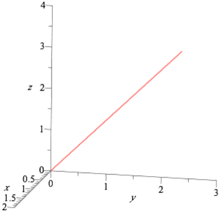

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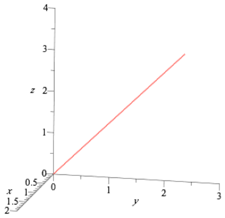
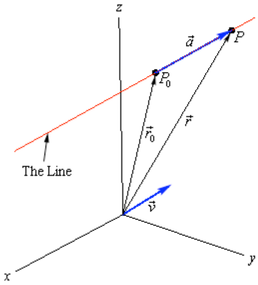
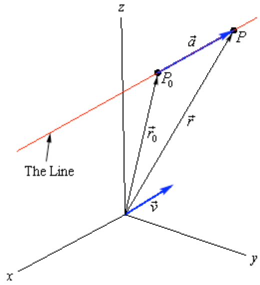


1. A point (x_0, y_0, z_0) which the line passes through
2. A direction $\vec{v}$ to which the line is parallel


z
$\vec{a}$
P
P_0
$\vec{r}$
$\vec{r}_0$
The Line
$\vec{v}$
x
y

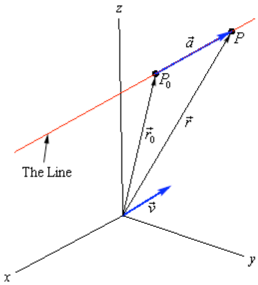


Let $P(x, y, z)$ be an arbitrary point on this line. Observe the three vectors

$$\vec{a} := \overrightarrow{P_0P} \qquad \text{the vector from } P_0 \text{ to } P$$

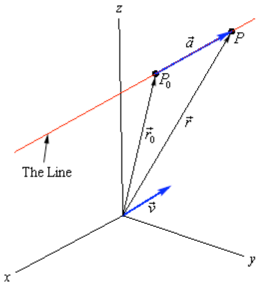
$$\vec{r_0} := \overrightarrow{OP_0} \qquad \text{the position vector of } P_0$$

$$\vec{r} := \overrightarrow{OP} \qquad \text{the position vector of } P$$



By vector addition,

$$\vec{r} = \vec{r_0} + \vec{a}$$



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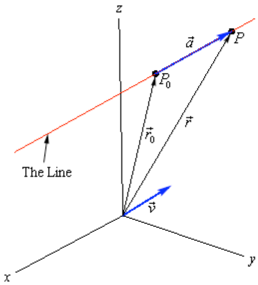

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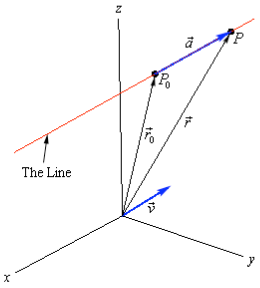

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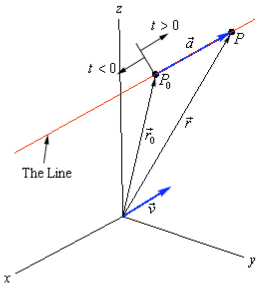
$\vec{r} = \vec{r_0} + t\vec{v}$ **the vector equation** of the line



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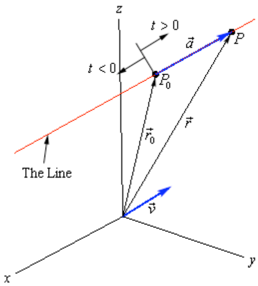

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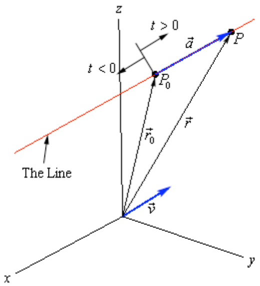
Write each vector in the component form:



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Write each vector in the component form: $\vec{r} =< x, y, z >$, $\vec{r_0} =< x_0, y_0, z_0 >$, $\vec{v} =< a, b, c >$

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

the parametric equations of the line.

From $x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$

If $a \neq 0, b \neq 0, c \neq 0$ and solve for t, we can get

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

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The parametric equations

$$x = 2 - 5t \qquad y = 4 - 3t \qquad z = 3 - 2t$$

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Note: The equations of a line are not unique.

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Given two different lines

$$L_1 : \vec{r} = \vec{r}_1 + t\vec{v}_1$$
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How many ways can two lines interact in 3D?

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When are two lines orthogonal?

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When are two lines orthogonal? $\vec{v}_1 \cdot \vec{v}_2 = 0$

Example 2 Determine whether the line L_1 and L_2 are parallel, skew or intersecting. If intersecting, find the point of intersection.

$$L_1 : x = 1 + 2t, y = 3t, z = 2 - t$$
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Solution: The direction vector of L_1 is $\vec{v}_1 = \; < 2, 3, -1 >$. The direction vector of L_2 is $\vec{v}_2 = \; < 1, 1, 3 >$. $\vec{v}_1$ and $\vec{v}_2$ are not scalar multiples of each other. So NOT parallel.

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$$x = 1 + 2t = -1 + s \Rightarrow 2t + 2 = s \qquad (\bigstar)$$

$$y = 3t = 4 + s \Rightarrow 3t - 4 = s \qquad (\bigstar\bigstar)$$

$$z = 2 - t = 1 + 3s \Rightarrow \frac{1}{3}(1 - t) = s \qquad (\bigstar\bigstar\bigstar)$$

Solve $(\bigstar)$ and $(\bigstar\bigstar)$. We get $t = 6$ and $s = 14$.

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Hence, L_1 and L_2 are skew.

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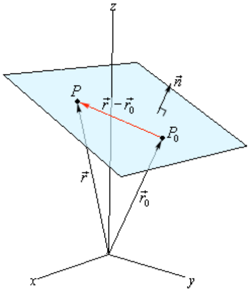
What to we need to define the equation of a plane?

1. A point in the plane, say $P_0(x_0, y_0, z_0)$
2. A vector orthogonal to the plane, say $\vec{n}$, called **normal vector**



For any point $P(x, y, z)$ in the plane, observe the three vectors

- $\overrightarrow{OP_0} := \vec{r}_0$, the position vector of point P_0
- $\overrightarrow{OP} := \vec{r}$, the position vector of point P
- $\overrightarrow{P_0P} = \vec{r} - \vec{r}_0$, a vector in the plane

Since $\vec{n}$ is orthogonal to the plane, it is orthogonal to any vector in the plane. Hence

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If write the vectors in the component way, $\vec{n} =< a, b, c >$, $\vec{r}_0 =< x_0, y_0, z_0 >$ and $\vec{r} =< x, y, z >$, then

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called a **scalar equation of the plane through** $P_0(x_0, y_0, z_0)$ **with normal vector** $\vec{n} =< a, b, c >$.

We can also write $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ as

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We can see that a linear equation in 3D represent a plane.

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$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & 2 \\ -4 & -1 & 1 \end{vmatrix} = 5\vec{i} - 13\vec{j} + 7\vec{k}$$

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Given two distinct planes

$$\gamma_1 : a_1x + b_1y + c_1z + d_1 = 0$$
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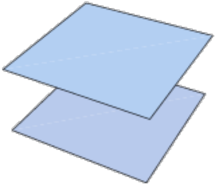

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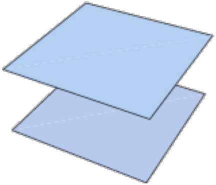
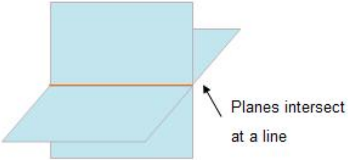


- **Intersecting** in a line, if not parallel



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Hence, we may let

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\vec{i} - 2\vec{k}$$

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Since the y component of the direction vector is 0. The symmetric equations of L are

$$\frac{x}{2} = \frac{z-1}{-2}, \quad y = 0$$

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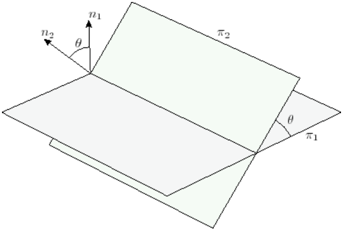

The angle θ between two planes can be defined as the angle between the two normal vectors, $\vec{n}_1$ and $\vec{n}_2$. And we have know

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}, \quad 0 \le \theta \le \frac{\pi}{2}$$

Example 4 (cont'd) Find the angle the two planes $x + y + z = 1$ and $x - y + z = 1$.

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Solution: $\vec{n}_1 =< 1, 1, 1 >$ and $\vec{n}_2 =< 1, -1, 1 >$. So

$$|\vec{n}_1| = \sqrt{3} \qquad |\vec{n}_2| = \sqrt{3} \qquad \vec{n}_1 \cdot \vec{n}_2 = 1$$

Hence

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \frac{1}{3} \Rightarrow \theta = \arccos(\frac{1}{3})$$

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1. From P_1 draw a line perpendicular to the plane. Let P_1' be the intersection. Then the distance is just $|P_1 P_1'|$.
2. Choose a point $P_0(x_0, y_0, z_0)$ in the plane, other than P_1'.
3. In the right triangle $\triangle P_1 P_1' P_0$, the distance is the scalar projection of $\overrightarrow{P_0 P_1}$ onto $\vec{n}$. So

$$
\begin{aligned}
D = |\text{comp}_{\vec{n}} \overrightarrow{P_0 P_1}|] &= \frac{|\overrightarrow{P_0 P_1} \cdot \vec{n}|}{|\vec{n}|} \\
&= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}
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2. The distance between the two planes is equal to the distance from P to $4x - 6y + 2z = 3$.

$$D = \frac{|4 \cdot 0 - 6 \cdot 0 + 2 \cdot 4 - 3|}{\sqrt{16 + 36 + 4}} = \frac{5}{2\sqrt{14}}$$

Example 6 In Example 2, we have found that
$L_1 : x = 1 + 2t, y = 3t, z = 2 - t$ and
$L_2 : x = -1 + s, y = 4 + s, z = 1 + 3s$ are skew. Find the distance between them.

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Choose a point P_2 on L_2 by letting $s = 0$, $P_2 = (-1, 4, 1)$. Then Γ_2 has an equation

$$10(x + 1) - 7(y - 4) - (z - 1) = 0 \Rightarrow 10x - 7y - z + 39 = 0$$

Choose a point P_1 on L_1 by letting $t = 0$, $P_1 = (1, 0, 2)$. Then the distance from P_1 to Γ_2 is

$$D = \frac{|10 \cdot 1 - 7 \cdot 0 - 1 \cdot 2 + 39|}{\sqrt{100 + 49 + 1}} = \frac{47}{5\sqrt{6}}$$