MATH 241 – Course outline

Chapter 1

Derivation of the heat equation for a thin rod:

Thermal energy: $c(x)\rho(x)A(x)u(x,t)\,dx$ (where c is specific heat, ρ is density, A is cross-sectional area)is thermal energy in a little bit of the rod.

Heat flux: $\varphi(x,t)$ per unit area (from left to right), so heat energy per unit time into the little bit is $\varphi(x,t)A(x) - \varphi(x+dx)A(x+dx)$.

Sources: $Q(x,t)A(x)dx$ is heat energy generated per unit time in the little bit.

Conservation of energy:

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \varphi}{\partial x} + Q(x,t).$$

Fourier's law: Assume $\varphi = -K_0(x)(\partial u/\partial x)$.

Altogether, as bad as it can get:

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + Q(x,t).$$

If c, ρ, K_0 are constants and $Q = 0$, set $k = K_0/(s\rho)$ and get usual heat equation $u_t = ku_{xx}$.

Boundary conditions: Dirichlet conditions give value of temperature at endpoints, e.g., $u(0,t) = a(t)$, $u(L,t) = b(t)$. Could also give (directly or indirectly) values of u_x at endpoints. (Insulated endpoints: $u_x = 0$ at ends).

Diffusion equation:

Density: $u(x,t)$, so total amount of chemical is $\int_a^b u(x,t)A(x)\,dx$.
Flux: $\varphi(x,t)$ amount moving to right per unit area per unit time, so

$$\frac{d}{dt}\int_a^b u(x,t)A(x)\,dx = \varphi(a,t)A(a) - \varphi(b,t)A(b).$$

Conservation law (continuity equation): If $A(x)$ is constant then $\dfrac{\partial u}{\partial t} + \dfrac{\partial \varphi}{\partial x} = 0$.

Fick's law of diffusion: Assume $\varphi(x) = -k\dfrac{\partial u}{\partial x}$.

Put last two together to get diffusion equation: $u_t = ku_{xx}$.

Boundary conditions: Dirichlet conditions give density u at endpoints. Can also give flux at endpoints which is $-ku_x$.

Equilibrium: u does not depend on t. Simplifies heat/diffusion equations to ODEs which are (usually) easily solved together with boundary values. Note: insulated boundaries $u_x = 0$ at both ends and no sources imply total heat/chemical is constant, so $\int u\,dx$ is equal to its initial value for all t.

Higher dimensions: replace u_{xx} with $\nabla^2 u$. In the plane

$$\nabla^2 u = u_{xx} + u_{yy} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial u^2}{\partial r^2}.$$

In three dimensions,

$$\begin{aligned}\nabla^2 u &= u_{xx} + u_{yy} + u_{zz}\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial u^2}{\partial r^2} + \frac{\partial^2 u}{\partial z^2}\\ &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2}.\end{aligned}$$

(note, if $K_0(\mathbf{x})$ is not constant, then instead of $\nabla^2 u$, get $\nabla\cdot(K_0(\mathbf{x})\nabla u)$).

Chapter 2

Separation of variables: For $u_t = ku_{xx}$ (or even a little more exotic) assume $u(x,t) = X(x)T(t)$, separate and get eigenvalue problem for X – depending on boundary conditions

$$X(x) = \sin\left(\frac{n\pi x}{L}\right),\ n = 1,2,\ldots \quad \lambda = \frac{n^2\pi^2}{L^2} \quad \text{if } X(0) = X(L) = 0.$$

$$X(x) = \sin\left(\frac{(n+\frac{1}{2})\pi x}{L}\right) \quad n = 0,1,2,\ldots \quad \lambda = \frac{(n+\frac{1}{2})^2\pi^2}{L^2} \quad \text{if } X(0) = X'(L) = 0.$$

$$X(x) = \cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right) \quad n = 0,1,2\ldots \quad \lambda = \frac{(n+\frac{1}{2})^2\pi^2}{L^2} \quad \text{if } X'(0) = X(L) = 0.$$

$$X(x) = \cos\left(\frac{n\pi x}{L}\right) \quad n = 0,1,2,\ldots \quad \lambda = \frac{n^2\pi^2}{L^2} \quad \text{if } X'(0) = X'(L) = 0.$$

Note particularly for the last one that $X(x) = 1$ is the solution for $n = 0$. Don't forget this term!

The corresponding t solutions are $T = e^{-\lambda kt}$ so get series, e.g.,

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 kt/L^2}\sin\left(\frac{n\pi x}{L}\right)$$

where a_n is determined by initial conditions $u(x,0) = f(x)$:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

This last formula must be adjusted according to which eigenfunctions are used (i.e., integrate f against the corresponding eigenfunction – for the $\lambda = 0$ eigenfunction 1 in the fourth case, the coefficient in front of integral becomes $1/L$.

Laplace equation (equilibrium solutions) on rectangles and disks:

Rectangles. Can only solve problems which have zero conditions (either value of u or normal derivative) on three sides and non-zero on the fourth. Then get sines or cosines in the direction where there are two zero sides (depending on boundary conditions as above) and sinh or cosh in the direction where one side is non-zero (depending on type of condition on the zero side). Example:

For $u(0,y) = u_x(L,y) = u(x,0) = 0$ and $u(x,H) = f(x)$ the x-direction has two zeros, but at L it is u_x which is zero so

$$u(x,y) = \sum_{n=0}^{\infty} a_n \sin\left(\frac{(n+\frac{1}{2})\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right) \qquad \text{where} \quad a_n = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f(x) \sin\left(\frac{(n+\frac{1}{2})\pi x}{L}\right) dx.$$

Another example: for $u_x(0,y) = g(y)$, $u_x(L,y) = 0$, $u(x,0) = 0$, $u(x,H) = 0$, the y direction has two zeros, so

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{H}\right) \cosh\left(\frac{n\pi(L-x)}{H}\right) \qquad \text{where} \quad b_n = \frac{2}{H\cosh\left(\frac{n\pi L}{H}\right)} \int_0^H g(x) \sin\left(\frac{n\pi y}{H}\right) dx,$$

and so forth.

Integrability condition: If normal derivatives are prescribed all the way around the boundary of the region, then they must integrate to zero.

Disks: Separating variables in Laplace equation in polar coordinates gives solution of the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

where a_0, a_n, b_n given by usual Fourier series formula for 2π-periodic functions. If half-disk or other wedge, then boundary conditions for θ come from radial edges. Watch out for integrability condition if normal derivative is prescribed all the way around.

Chapter 3

"Full" Fourier series: Given $f(x)$ for $-L < x < L$, its full Fourier series is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Assuming f is piecewise smooth, the series converges to the periodic extension of f (with period $2L$) except where this function has jumps, and to the average of the left and right limits at the jumps. So in general it converges to f only (mostly) on $-L < x < L$.

Fourier sine series: Given $f(x)$ for $0 < x < L$, its Fourier sine series is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

which converges to the odd periodic extension of f (with period $2L$) except at discontinuities etc..

Fourier cosine series: Given $f(x)$ for $0 < x < L$, its Fourier cosine series is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{where} \quad a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

which converges to the even periodic extension of f (with period $2L$) except at discontinuities etc..

Complex Fourier series: Given $f(x)$ for $-L < x < L$ its complex Fourier series is

$$\sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L} \quad \text{where} c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx.$$

If a Fourier series converges to a continuous function, and if the derivative of the function is piecewise smooth, then you can differentiate the Fourier series term by term (or differentiate a sine series to get a cosine series and vice versa).

You can integrate a Fourier series (sine series, cosine series) term by term but you might have to be careful about the constant of integration and the integral of the constant term if there is one (which would give $a_0 x$ which is not strictly speaking part of a Fourier series).

Chapter 4

Derivation of the wave equation: Based on $F = ma$ – String describes a curve given by $u(x,t)$ where $y = u(x,t)$ for fixed t is the shape of the curve at time t. The force in the string is tension with magnitude $T(x)$ tangent to the string. Mass of a little bit of string is $\rho(x)\Delta x$. Use approximation $\sin\theta \approx \tan\theta$ for small values of θ (the angle the string makes with the x-axis, so $\tan\theta = u_x$). The component of tension forces along the string cancels, so string accelerates only perpendicular to x-axis, and

$$ma \approx \rho(x)\Delta x u_{tt}(x,t) \approx T(x+\Delta x)\tan\theta(x+\Delta x) - T(x)\tan\theta(x) = T(x+\Delta x)u_x(x+\Delta x,t) - T(x)u_x(x,t).$$

Divide by Δx and let $\Delta x \to 0$ and get

$$\rho(x)u_{tt} = (T(x)u_x)_x$$

(plus $\rho(x)Q(x,t)$ if there is an external force Q acting on the string, e.g., gravity would have $Q = -g$ etc). If ρ and T are constant and $Q = 0$ we get familiar wave equation $u_{tt} = c^2 u_{xx}$ with $c^2 = T/\rho$.

Initial and boundary conditions: Since the wave equation is about acceleration, need initial position and velocity ($u(x,0)$ and $u_t(x,0)$). At endpoints, prescribe position of string (Dirichlet conditions) $u(0,t)$ and $u(L,t)$ or derivatives or some combination.

Separation of variables: Like heat equation except this time t solutions are sines and cosines and (if $u(0,t) = u(L,t) = 0$ say):

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right].$$

The a_n are determined by $u(x,0) = f(x)$, and b_n are determined by $u_t(x,0) = g(x)$ after differentiating with respect to t and putting $t = 0$, so

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \quad \text{and} \quad b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Time (circular) frequencies of vibrations are $n\pi c/L$.

Chapter 5

Sturm-Liouville problems: In separation of variables, we need to find eigenfunctions and eigenvalues of boundary value problems for second-order ODEs. Get a good theory for problems in Sturm-Liouville form:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda w(x)y = 0$$

for $a \leqslant x \leqslant b$ with two boundary conditions on some combinations of $y(a), y(b), y'(a), y'(b)$ (e.g., $y(a) = 0$ and $y'(b) = 0$ etc). The problem is called "regular" if $p(x)$ and $w(x)$ are positive on the entire interval of consideration.

Eigenfunctions of Sturm-Liouville problems: Many problems can be put into this form after multiplication by a function. Such a problem has an infinite sequence of eigenvalues $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ and corresponding eigenfunctions $\varphi_1(x), \varphi_2(x), \ldots$. Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $w(x)$, so

$$\langle \varphi_m(x)\,,\,\varphi_n(x)\rangle = \int_a^b \varphi_m(x)\varphi_n(x)w(x)\,dx = 0$$

and "any" function on $[a,b]$ can be expanded in a Fourier-like series:

$$f(x) = \sum_{n=1}^{\infty} a_n\varphi_n(x) \quad \text{with} \quad a_n = \frac{\langle f\,,\,\varphi_n\rangle}{\langle \varphi_n\,,\,\varphi_n\rangle}.$$

Chapter 7

Higher-dimensional wave and heat equation problems: For $u_t = k\nabla^2 u$ or $\mathbf{u}_{tt} = c^2\nabla^2 u$ with Dirichlet or Neumann (normal derivative = 0) boundary conditions, we separate out the time variable and get

$$\frac{T'(t)}{kT(t)} = \frac{\nabla^2 Z(\mathbf{x})}{Z(\mathbf{x})} = -\lambda \quad \text{or} \quad \frac{T''(t)}{c^2T(t)} = \frac{\nabla^2 Z(\mathbf{x})}{Z(\mathbf{x})} = -\lambda$$

for $t > 0$ and $\mathbf{x}$ in the 2 or 3-dimensional domain. The solution will be

$$u(\mathbf{x},t) = \sum_{n=1}^{\infty} a_n e^{-k\lambda_n t}\varphi_n(\mathbf{x}) \quad \text{or} \quad u(\mathbf{x},t) = \sum_{n=1}^{\infty} [a_n \cos(c\sqrt{\lambda_n}\, t) + b_n \sin(c\sqrt{\lambda_n}\, t)]\varphi_n(\mathbf{x})$$

where $\nabla^2\varphi_n(\mathbf{x}) + \lambda_n\varphi_n(\mathbf{x}) = 0$. The circular frequencis for the wave equation are $c\sqrt{\lambda_n}$. Three main examples:

Rectangles and boxes: Say $0 < x < L$, $0 < y < H$ (could have z as well), if Dirichlet conditions on edges, get

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn} e^{-k(n^2+m^2)t} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) \quad \text{where} \quad a_{mn} = \frac{4}{HL}\int_0^H\int_0^L f(x,y)\sin\left(\frac{m\pi x}{L}\right)\,dx\,dy$$

for the heat equation with $u(x,y,0) = f(x,y)$ and something similar for the wave equation except with sines and cosines of t.

Disks: This time the eigenfunctions are of the form $\varphi_{mn}(r,\theta) = J_m(z_{mn}r)\cos m\theta$ for $m = 0,1,2,\ldots$ and $J_m(z_{mn}r)\sin m\theta$ for $m = 1,2,3,\ldots$, where J_m is the Bessel function (of the first kind) of order m and z_{mn} is the nth positive zero of J_m. The corresponding eigenvalues are z_{mn}^2. So the solution is

$$\begin{aligned} u(x,y,t) = &\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(z_{mn}r)\cos m\theta[a_{mn}\cos(cz_{mn}t) + b_{mn}\sin(cz_{mn}t)] \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(z_{mn}r)\sin m\theta[c_{mn}\cos(cz_{mn}t) + d_{mn}\sin(cz_{mn}t)], \end{aligned}$$

where a_{mn} and c_{mn} are determined from the initial position and b_{mn} and d_{mn} are determined from the initial velocity. The circular frequencies are cz_{mn}.

If the initial data are circularly symmetric (i.e., do not depend on θ), then only the $m = 0$ terms appear (so only J_0).

Cylinders: Equilibrium (Laplace equation): The solution of the problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

for $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant z \leqslant H$ together with boundary conditions

$$\begin{aligned} u(r,\theta,H) &= f(r,\theta) \quad \text{on the top} \\ u(r,\theta,0) &= g(r,\theta) \quad \text{on the bottom} \\ u(a,\theta,z) &= h(\theta,z) \quad \text{on the side} \end{aligned}$$

is

$$u(r,\theta,z) = u_1(r,\theta,z) + u_2(r,\theta,z) + u_3(r,\theta,z),$$

where (with z_{nm} being the mth positive zero of the Bessel function $J_n(x)$)

$$u_1(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}r}{a}\right)\sinh\left(\frac{z_{nm}z}{a}\right)[a_{nm}\cos n\theta + b_{nm}\sin n\theta],$$

$$u_2(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}r}{a}\right)\sinh\left(\frac{z_{nm}(H-z)}{a}\right)[c_{nm}\cos n\theta + d_{nm}\sin n\theta],$$

and

$$u_3(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} I_n\left(\frac{m\pi r}{H}\right)\sin\left(\frac{m\pi z}{H}\right)[e_{nm}\cos n\theta + f_{nm}\sin n\theta].$$

The coefficients a_{nm}, b_{nm}, c_{nm}, d_{nm}, e_{nm} and f_{nm} are given by

$$a_{0m} = \frac{\displaystyle\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_0\left(\frac{z_{0m}r}{a}\right)\,dr\,d\theta}{\pi a^2 J_1(z_{0m})^2 \sinh\left(\dfrac{z_{0m}H}{a}\right)} \quad \text{for } n=0, m\geqslant 1$$

$$a_{nm} = \frac{2\displaystyle\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\cos n\theta\,dr\,d\theta}{\pi a^2 J_{n+1}(z_{nm})^2 \sinh\left(\dfrac{z_{nm}H}{a}\right)} \quad \text{for } n\geqslant 0, m\geqslant 1$$

$$b_{nm} = \frac{2\displaystyle\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\sin n\theta\,dr\,d\theta}{\pi a^2 J_{n+1}(z_{nm})^2 \sinh\left(\dfrac{z_{nm}H}{a}\right)} \quad \text{for } n\geqslant 0, m\geqslant 1$$

$$c_{0m} = \frac{\displaystyle\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_0\left(\frac{z_{0m}r}{a}\right)\,dr\,d\theta}{\pi a^2 J_1(z_{0m})^2 \sinh\left(\dfrac{z_{0m}H}{a}\right)} \quad \text{for } n=0, m\geqslant 1$$

$$c_{nm} = \frac{2\displaystyle\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\cos n\theta\,dr\,d\theta}{\pi a^2 J_{n+1}(z_{nm})^2 \sinh\left(\dfrac{z_{nm}H}{a}\right)} \quad \text{for } n\geqslant 1, m\geqslant 1$$

$$d_{nm} = \frac{2\displaystyle\int_{-\pi}^{\pi}\int_0^a rg(r,\theta)J_n\left(\frac{z_{nm}r}{a}\right)\sin n\theta\,dr\,d\theta}{\pi a^2 J_{n+1}(z_{nm})^2 \sinh\left(\dfrac{z_{nm}H}{a}\right)} \quad \text{for } n\geqslant 1, m\geqslant 1$$

$$e_{0m} = \frac{\displaystyle\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\,dz\,d\theta}{\pi H I_0\left(\dfrac{m\pi a}{H}\right)} \quad \text{for } n=0, m\geqslant 1$$

$$e_{nm} = \frac{2\displaystyle\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\cos n\theta\,dz\,d\theta}{\pi H I_n\left(\dfrac{m\pi a}{H}\right)} \quad \text{for } n\geqslant 1, m\geqslant 1$$

$$f_{nm} = \frac{2\displaystyle\int_{-\pi}^{\pi}\int_0^H h(\theta,z)\sin\left(\frac{m\pi z}{H}\right)\sin n\theta\,dz\,d\theta}{\pi H I_n\left(\dfrac{m\pi a}{H}\right)} \quad \text{for } n\geqslant 1, m\geqslant 1.$$

In u_3 and its coefficients, I_n is the nth modified Bessel function (hyperbolic Bessel function). For the corresponding heat and wave equations, you can figure out the eigenfunctions of ∇^2 which are similar to the pieces of these series (but no I_n, only J_n).

Balls: The spherical Laplace equation has solution

$$u(\rho,\varphi,\theta) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\rho^n[a_{mn}\cos m\theta + b_{mn}\sin m\theta]P_n^m(\cos\varphi)$$

where $P_n^m(x)$ is the associated Legendre function

$$P_n^m(x) = (x^2-1)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

where $P_n(x) = P_n^0(x)$ is the nth Legendre polynomial. (The first few P_n are $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$, $P_3(x)=\frac{1}{2}(5x^3-3x)$, etc.. They are orthogonal on $[-1,1]$ with respect to $w(x)=1$. If the boundary data are independent of θ, then only the $P_n^0(\cos\varphi) = P_n(\cos\varphi)$ terms appear, no associated ones.

Solutions of the heat and wave equations have terms like

$$e^{-kz_{n\ell}^2 t/R^2}\frac{J_{n+\frac{1}{2}}\left(\dfrac{z_{n\ell}\rho}{R}\right)}{\sqrt{\rho}}\cos m\theta P_n^m(\cos\varphi) \quad \text{and} \quad \cos\left(\frac{cz_{n\ell}t}{R}\right)\frac{J_{n+\frac{1}{2}}\left(\dfrac{z_{n\ell}\rho}{R}\right)}{\sqrt{\rho}}\cos m\theta P_n^m(\cos\varphi)$$

where the $\cos m\theta$ could be $\sin m\theta$ and the $\cos(cz_nt/R)$ could be $\sin(cz_nt/R)$, R is the radius of the sphere, $z_{n\ell}$ is the ℓth positive zero of $J_{n+\frac{1}{2}}$. Circular frequencies for the wave equation are $cz_{n\ell}/R$. (Note: in the $n=0$ term, $J_{\frac{1}{2}}(x) = \sin x/\sqrt{x}$ so $z_{0\ell} = \ell\pi$.)

Chapter 8

Inhomogeneous equations and boundary conditions: If a problem for the wave or heat (or related) equation for $u(x,t)$ has an inhomogeneous source or boundary condition, first consider whether there is an equilibrium solution $u_{\text{eq}}(x)$, and subtract it from $u(x,t)$ – set $v(x,t) = u(x,t) - u_{\text{eq}}(x)$ and get a simpler equation for v. If there's no equilibrium solution, find a relatively simple function $u_{\text{bd}}(x,t)$ that satisfies the inhomogeneous boundary conditions, and subtract that from $u(x,t)$ and so get a problem for $v(x,t)$ that has homogeneous boundary conditions.

The second step is to seek $v(x,t)$ in the form

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x)$$

where $\varphi_n(x)$ is the nth eigenfunction of the x-part of the differential equation and get and solve differential equations for $a_n(t)$. Initial conditions for $a_n(0)$ will usually involve finding Fourier series for the initial data of v.

Chapter 10

Fourier transforms:

$$\widehat{f}(\omega) = F(\omega) = \mathcal{F}[f(x)](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ix\omega}\,dx$$

and

$$\check{F}(x) = f(x) = \mathcal{F}^{-1}[F(\omega)](x) = \int_{-\infty}^{\infty} F(\omega)e^{-ix\omega}\,d\omega.$$

Properties:

1. **Linearity**: $\mathcal{F}\left[\alpha f(x) + \beta g(x)\right] = \alpha\mathcal{F}\left[f(x)\right] + \beta\mathcal{F}\left[g(x)\right]$.
2. **Translation** (or shifting): $\mathcal{F}\left[f(x-a)\right](\omega) = e^{i\omega a}\mathcal{F}\left[f(x)\right](\omega)$. And in the other direction, $\mathcal{F}\left[e^{iax}f(x)\right](\omega) = \mathcal{F}\left[f(x)\right](\omega + a)$.
3. **Scaling**: $\mathcal{F}\left[\frac{1}{a}f\left(\frac{x}{a}\right)\right](\omega) = \mathcal{F}\left[f(x)\right](a\omega)$, and likewise $\mathcal{F}\left[f(ax)\right](\omega) = \frac{1}{a}\mathcal{F}\left[f(x)\right]\left(\frac{\omega}{a}\right)$.
4. **Operational property** (derivatives): $\mathcal{F}\left[f'(x)\right](\omega) = -i\omega\mathcal{F}\left[f(x)\right](\omega)$, and $\mathcal{F}\left[xf(x)\right](\omega) = -i\frac{d}{d\omega}\left(\mathcal{F}\left[f(x)\right](\omega)\right)$.
5. $\mathcal{F}\left[f(x)\right](\omega) = \frac{1}{2\pi}\mathcal{F}^{-1}\left[f(-y)\right](\omega)$.
6. $\mathcal{F}^{-1}\left[F(\omega)\right](x) = 2\pi\mathcal{F}\left[F(-\alpha)\right](x)$
7. Convolutions:

   $$\mathcal{F}\left[f * g\right](\omega) = 2\pi\mathcal{F}\left[f\right](\omega)\mathcal{F}\left[g\right](\omega) \quad \text{or} \quad \widehat{(f * g)}(\omega) = 2\pi\widehat{f}(\omega)\widehat{g}(\omega)$$

   and

   $$\mathcal{F}^{-1}\left[F * G\right](x) = \check{F}(x)\check{G}(x)$$

8. Parseval:

   $$\langle f\,,\,g\rangle = 2\pi\left\langle \widehat{f}\,,\,\widehat{g}\right\rangle$$

   and

   $$\|f\|^2 = 2\pi\|\widehat{f}\|^2.$$

Examples:

1. If
$$S_a(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{otherwise} \end{cases}$$
then
$$\mathcal{F}\left[S_a(x)\right](\omega) = \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega x}\,dx = \frac{e^{i\omega a} - e^{-i\omega a}}{2\pi i\omega} = \frac{\sin a\omega}{\pi\omega}$$
and
$$\mathcal{F}\left[\frac{\sin ax}{\pi x}\right](\omega) = \frac{1}{2\pi}S_a(\omega).$$

2.
$$\mathcal{F}\left[e^{-ax^2/2}\right](\omega) = \frac{1}{\sqrt{2\pi a}}e^{-\omega^2/(2a)}$$
In particular,
$$\mathcal{F}\left[e^{-x^2/2}\right] = \frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}.$$

3.
$$\mathcal{F}\left[e^{-a|x|}\right](\omega) = \frac{a}{\pi(a^2+\omega^2)}$$
and
$$\mathcal{F}\left[\frac{1}{a^2+x^2}\right] = \frac{1}{2a}e^{-a|\omega|}.$$

To solve PDE problems where one of the variables (usually x) goes from $-\infty$ to ∞, take the Fourier transform of everything in that variable (could do it in more than one variable) – doing this usually gets rid of differentiation in that variable (so can convert PDE to ODE or from ODE to algebra). Solve the resulting simplified equation for $\widehat{u}$ and attempt to do inverse transform to get u. Two important formulas to come from this:

Solution of initial value problem for heat equation: $u_t = ku_{xx}$ with $u(x,0) = f(x)$ Solution is
$$f * G(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4kt}\,dy$$
$G(x,t)$ is the fundamental solution of the heat equation (or heat kernel).

D'Alembert's solution of initial value problem for the wave equation: $u_{tt} = c^2u_{xx}$ with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Solution is
$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du.$$