# Fall 2013

Name:

Recitation Number and Day/Time:

---

Please *turn off and put away all electronic devices*. You may use both sides of a $8.5'' \times 11''$ sheet of paper for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**. Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good luck!

---

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

Your signature

---

| Question Number | Points Possible | Your Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |

| Midterm 1 Total Score | /100 |
|---|---|

# A Partial Table of Integrals

$$\int_0^x u\cos nu\ du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\ du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\ du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\ du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\ du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\ du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\ du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

# Laplacian in polar coordinates

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

**(1)** [20 POINTS] The temperature of a rod is described by the following equations:

$$\left|\begin{array}{l} u_t = u_{xx} + e^{-x}, \quad 0 \le x \le 1, \quad t \ge 0 \\ u(0,t) + 2u_x(0,t) = 0 \\ u_x(1,t) = 3 \\ u(x,0) = \sin x \end{array}\right.$$

When it reaches equilibrium, what is the temperature at $x = 0$?

(A) $0$ (B) $-8 + 2e^{-1}$ (C) $-e^{-1}$

(D) $\sin(1) - e^{-1}$ (E) $\sin(1)$ (F) $3 + e^{-1}$

---

Answer: The equilibrium temperature $u(x)$ satisfies

$$u''(x) + e^{-x} = 0, \ 0 \le x \le 1$$

and the boundary conditions

$$u(0) + 2u'(0) = 0, \ u(1) = 3.$$

Integrating twice, we get

$$u(x) = -e^{-x} + c_1 x + c_2,$$

where $c_1$ and $c_2$ are arbitrary constants. The boundary conditions gives

$$-1 + c_2 + 2(1 + c_1) = 0$$

and

$$e^{-1} + c_1 = 3,$$

so $c_1 = 3 - e^{-1}$, $c_2 = -7 + 2e^{-1}$, and

$$u(x) = -e^{-x} + (3 - e^{-1})x + (-7 + 2e^{-1}).$$

Plug in $x = 0$, we get

$$u(0) = -1 + (-7 + 2e^{-1}) = -8 + 2e^{-1}.$$

The correct answer is (B).

**(2)** 20 POINTS Let $u(x,t)$ be the solution of the equation

$$u_t = 3u_{xx}, \quad 0 \le x \le 3, \quad t \ge 0$$

satisfying the boundary conditions

$$u_x(0,t) = 0 \quad \text{and} \quad u_x(3,t) = 0$$

$$u(x,0) = 3 - \cos(3\pi x)$$

Compute $u\left(\frac{1}{2}, 2\right)$.

(A) 0 (B) $\pi$ (C) $3 - e^{-54\pi^2}$

(D) $3 - e^{-162\pi^2}$ (E) 3 (F) $-e^{-162\pi^2}$

---

Answer: We use separation of variables, let

$$u(x,t) = \phi(x)G(t)$$

Substituting in the equation we get

$$\frac{dG}{dt}\phi = 3G\frac{d^2\phi}{dx^2},$$

so we will have

$$\frac{1}{3G}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

for some constant $\lambda$.

The boundary conditions imply $\phi'(0) = 0$ and $\phi'(3) = 0$, so $\phi$ must solve

$$\phi'' + \lambda\phi = 0, \quad \phi'(0) = 0, \quad \phi'(3) = 0$$

hence

$$\lambda_n = \left(\frac{n\pi}{3}\right)^2, \; n = 0, 1, 2, \cdots$$

with the corresponding eigenfunction $\phi_n$ given by

$$\phi_n(x) = c \cdot \cos\frac{n\pi x}{3}$$

and for each eigenvalue $\lambda_n$, $G_n(t)$ satisfies

$$G_n' = -3\lambda_n G_n$$

so

$$G_n(t) = c \cdot e^{-3(\frac{n\pi}{3})^2 t}$$

By the principle of superposition, we form the general solution

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{3} e^{-3(\frac{n\pi}{3})^2 t}$$

By initial condition $u(x,0) = 3-3\cos(3\pi x)$, we can identify $A_0 = 3$, $A_9 = -3$, while other coefficients are 0. So

$$u(x,t) = 3 - 3\cos(3\pi x)e^{-27\pi^2 t}$$

Plug in $x = \frac{1}{2}$, $t = 2$, we get

$$u(\frac{1}{2}, 2) = 3 - 3\cos\frac{3\pi}{2}e^{-54\pi^2} = 3.$$

So the correct answer is (E).

**(3)** 20 POINTS Let $f(x) = x^2 - 4x$ for $0 \le x \le 2$, and let

$$\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

its Fourier cosine series. What is the value of $a_4$?

(A) $\frac{1}{\pi^2}$ (B) $-\frac{1}{4\pi^3}$ (C) $2 - \frac{1}{4\pi^3}$

(D) $-\frac{1}{\pi^2} - \frac{1}{4\pi^3}$ (E) 3 (F) $\frac{1}{8\pi^3}$

---

Answer: From the formula of Fourier cosine coefficients we compute

$$\begin{aligned}
a_4 &= \frac{2}{2}\int_0^2 (x^2 - 4x)\cos\frac{4\pi x}{2}dx \\
&= \frac{1}{2\pi}\int_0^2 (x^2 - 4x)d(\sin 2\pi x) \\
&= \frac{1}{2\pi}(x^2 - 4x)\sin 2\pi x\Big|_0^2 - \frac{1}{2\pi}\int_0^2 \sin 2\pi x d(x^2 - 4x) \\
&= -\frac{1}{2\pi}\int_0^2 (2x - 4)\sin 2\pi x dx \\
&= \frac{1}{4\pi^2}\int_0^2 (2x - 4)d(\cos 2\pi x) \\
&= \frac{1}{4\pi^2}(2x - 4)\cos 2\pi x\Big|_0^2 - \frac{1}{4\pi^2}\int_0^2 \cos 2\pi x d(2x - 4) \\
&= \frac{1}{4\pi^2}(0 - (-4)) - \frac{1}{4\pi^2}\int_0^2 2\cos 2\pi x dx \\
&= \frac{1}{\pi^2} - \frac{1}{4\pi^2}\frac{2}{2\pi}\sin 2\pi x\Big|_0^2 \\
&= \frac{1}{\pi^2} - \frac{1}{4\pi^3}(\sin 4\pi - \sin 0) \\
&= \frac{1}{\pi^2}.
\end{aligned}$$

The correct answer is (A).

(4) 20 POINTS Solve the Laplace equation of $u(r,\theta)$ on a $90^o$ sector of a disk of radius 3:

$$\left|\begin{array}{c} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0,\ 0\le r\le 3,\quad 0\le\theta\le\frac{\pi}{2} \\ u(r,0)=0 \\ u\left(r,\frac{\pi}{2}\right)=0 \\ |u(0,\theta)|<+\infty \\ u(3,\theta)=1 \end{array}\right.$$

---

Answer: Use separation of variables in polar coordinates, set

$$u(r,\theta)=G(r)\phi(\theta),$$

plug in the equation and divide by $\frac{1}{r^2}G\phi$, we get

$$-\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)=\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}=-\lambda.$$

The boundary conditions $u(r,0)=0$ and $u(r,\frac{\pi}{2})=0$ implies $\phi(0)=0$ and $\phi(\frac{\pi}{2})=0$, so $\phi$ should solve the equation

$$\phi''+\lambda\phi=0$$

together with the boundary conditions

$$\phi(0)=0,\ \phi\left(\frac{\pi}{2}\right)=0.$$

We know the eigenvalues are

$$\lambda_n=4n^2,\ n=1,2,\cdots$$

and the corresponding eigenfunctions are

$$\phi_n(\theta)=c\cdot\sin 2n\theta.$$

Plug each eigenvalue to the equation of $G$:

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)=\lambda$$

we see that $G$ satisfies the Cauchy-Euler equation:

$$r(rG')'=4n^2G,\ n=1,2,\cdots$$

so

$$G_n(r)=c_1\cdot r^{2n}+c_2\cdot r^{-2n}.$$

Where $c_1$ and $c_2$ are two arbitrary constants. By $|u(0,\theta)<+\infty|$, solution has to be bounded at the origin, so $G$ cannot contain the $r^{-2n}$ term, therefore

$$G_n(r)=c\cdot r^{2n}.$$

By principle of superposition, the general solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n r^{2n} \sin 2n\theta.$$

The condition $u(3,\theta) = 1$ gives

$$\sum_{n=1}^{\infty} B_n 3^{2n} \sin 2n\theta = 1, \ \ 0 \le \theta \le \frac{\pi}{2},$$

we can compute $B_n 3^{2n}$ by the formula of Fourier sine coefficients

$$\begin{aligned} B_n 3^{2n} &= \frac{2}{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 1 \cdot \sin 2n\theta d\theta \\ &= \frac{4}{\pi}\Big( -\frac{1}{2n} \cos 2n\theta \Big)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{2}{n\pi}(1 - \cos n\pi) \\ &= \frac{2}{n\pi}(1 - (-1)^n) \end{aligned}$$

So

$$B_n = \frac{2}{3^{2n} n\pi}(1 - (-1)^n),$$

and

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{2}{3^{2n} n\pi}(1 - (-1)^n) r^{2n} \sin 2n\theta.$$

**(5)** 20 POINTS

(a) Compute the Fourier series of

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0, \\ x, & \text{if } 0 < x < \pi \end{cases}$$

on the interval $[-\pi, \pi]$. Fully simplify your answer - the formula for the coefficients should not contain any sines or cosines.

(b) What does this Fourier series converge to when $x = \pi$? Justify your answer.

---

Answer:

(a) Fourier series of $f(x)$ on the interval $[-\pi, \pi]$ is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{2\pi} \int_0^{\pi} x dx \\ &= \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi} \int_0^{\pi} x \cos nx dx \\ &= \frac{1}{\pi} \frac{\cos n\pi + n\pi \sin n\pi - 1}{n^2} \\ &= \frac{(-1)^n - 1}{n^2 \pi} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\ &= \frac{1}{\pi} \int_0^{\pi} x \sin nx dx \\ &= \frac{1}{\pi} \frac{\sin n\pi - n\pi \cos n\pi}{n^2} \\ &= \frac{(-1)^{n+1}}{n} \end{aligned}$$

So the Fourier series of $f(x)$ is

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \Big( \frac{(-1)^n - 1}{n^2 \pi} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \Big).$$

(b) By convergence theorem, denote $\tilde{f}$ as the periodic extension of $f$, then the Fourier series converges at $x = \pi$ to

$$\frac{\tilde{f}(\pi^-) + \tilde{f}(\pi^+)}{2} = \frac{0 + \pi}{2} = \frac{\pi}{2}.$$