Math 241-002 Midterm 2
Fall 2013

Name:

Recitation Number and Day/Time:

Please *turn off and put away all electronic devices.* You may use both sides of a $8.5'' \times 11''$ sheet of paper for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**. Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good luck!

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Your signature

Question Number	Points Possible	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	

Midterm 2 Total Score	/100

A Partial Table of Integrals

$$\int_0^x u\cos nu\ du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\ du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\ du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\ du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\ du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\ du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\ du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

Laplacian in polar coordinates

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(1) $\boxed{\text{20 POINTS}}$ Let $u(x,t)$ be a solution of the BVP

$$\left|\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < \pi,\ t > 0, \\ u(0,t) &= 1, \\ u(\pi,t) &= 2\pi + 1, \\ u(x,0) &= 1 - \sin(x), \\ u_t(x,0) &= 0. \end{aligned}\right.$$

Find the value of $u\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(A) $\frac{\pi}{2}$ (B) 0 (C) -1

(D) $\frac{1}{\pi}$ (E) $1+\pi$ (F) $\frac{1}{4} - \frac{3\pi}{2}$

Answer 1.

Since this equation has a non-homogeneous boundary condition, the solution will be equilibrium solution plus solution of a homogeneous problem.

Denote the equilibrium as $v(x)$, it satisfies

$$\left|\begin{aligned} 0 &= v''(x), \quad 0 < x < \pi \\ v(0) &= 1, \\ v(\pi) &= 2\pi + 1, \end{aligned}\right.$$

and we find $v(x) = 2x + 1$.

Now let $w(x,t) = u(x,t) - v(x)$, then w satisfies the homogeneous equation

$$\left|\begin{aligned} w_{tt} &= w_{xx}, \quad 0 < x < \pi,\ t > 0, \\ w(0,t) &= 0, \\ w(\pi,t) &= 0, \\ w(x,0) &= -\sin x - 2x, \\ w_t(x,0) &= 0. \end{aligned}\right.$$

and we get

$$w(x,t) = \sum_{n=1}^{\infty} \sin(nx)\big(a_n \cos(nt) + b_n \sin(nt)\big)$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} (-\sin x - 2x)\sin(nx)dx$$

and

$$b_n = 0$$

so

$$u(x,t) = v(x) + w(x,t) = 2x + 1 + \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nt)$$

and

$$u\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \pi + 1 + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{2} \cos\frac{n\pi}{2} = \pi + 1$$

The last identity holds because for any integer n, either $\sin\frac{n\pi}{2}$ or $\cos\frac{n\pi}{2}$ is zero. The correct answer is (E).

(2) 20 POINTS Let $u(x,t)$ be the vertical displacement of a vibrating string of infinite length. The string has constant density $\rho = 1$ and tension with constant magnitude $T = 9$. The initial position $u(x,0) = p(x)$ and velocity $u_t(x,0) = v(x)$ are given by and

$$p(x) = \begin{cases} 1 - |2x-1|, & \text{when } 0 < x < 1, \\ 0, & \text{when } x < 0 \text{ or } x > 1, \end{cases}$$

and $v(x) = 0$ for all x. Calculate $u\left(\frac{1}{2}, \frac{1}{2}\right)$.

(A) $\frac{3}{2}$ (B) 0 (C) $\frac{3}{4}$

(D) 1 (E) $\frac{1}{6}$ (F) $\frac{1}{2}$

Answer 2.

The equation of this vibrating string is

$$u_{tt} = 9u_{xx}$$

By d'Alembert's formula,

$$u(x,t) = \frac{1}{2}\big(p(x+3t) + p(x-3t)\big) + \frac{1}{6}\int_{x-3t}^{x+3t} v(s)ds$$

notice $v(x) = 0$, so

$$u(x,t) = \frac{1}{2}\big(p(x+3t) + p(x-3t)\big)$$

and

$$u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}\big(p(2) + p(-1)\big) = 0$$

So the correct answer is (B).

(3) 20 POINTS Consider the Sturm-Liouville equation

$$\phi'' - 7\phi + \lambda(x^2+2)\phi = 0$$

for a function $\phi(x)$ defined for $0 \le x \le 1$ with boundary conditions

$$\phi(0) = 0,$$
$$\phi'(1) = 0.$$

Let $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the set of of all eigenvalues of the above equation, and let $\phi_n(x)$ be an eigenfunction for the eigenvalue λ_n chosen so that $\int_0^1 \phi_n^2(x)(x^2+2)dx = 1$, $n \ge 1$. Which one of the following statements is true? Justify your reasoning.

(A) $\displaystyle\int_0^1 \phi_n^2(x)dx = 0$ for $n \ge 1$.

(B) $\phi_4(x)\phi_5(x) > 0$ for all $0 < x < 1$.

(C) this is a singular Sturm-Liouville BVP.

(D) $\displaystyle\lim_{n\to+\infty} \lambda_n = +\infty$.

(E) If $n \gg 0$, then $|\phi_n(x)| > 0$ for all $0 < x < 1$

(F) If $a_n = \int_0^1 (2x - x^2)\phi_n(x)(x^2+2)dx$, then

$$\sum_{n=1}^{\infty} a_n \phi_n\left(\frac{1}{2}\right)$$

converges to 2.

Answer 3.

(A) is not true because $\phi_n(x)$ is continuous and not identically zero for $n \ge 1$, and therefore it's square integration cannot be zero.
(B) is not true because $\phi_n(x)$ has $(n-1)$ zeros in $0 < x < 1$, and in particular, $\phi_4(x)\phi_5(x)$ has at least 5 zeros in the interval $(0,1)$, thus cannot be always positive.
(C) is not true because we can identify

$$p(x) = 1, \quad q(x) = -7, \quad \sigma(x) = x^2 + 2$$

all of them are continuous in the closed interval $[0,1]$, and $p(x) > 0$, $\sigma(x) > 0$, also the boundary conditions are homogeneous, hence this is a regular Sturm-Liouville BVP.
(D) is true because of Sturm-Liouville theorem.
(E) is not true because $|\phi_n(x)|$ has $(n-1)$ zeros in $(0,1)$ and cannot be always positive.
(F) is not true.Since $\frac{1}{2}$ is a continuous point of the function $2x - x^2$, according to Sturm-Liouville theorem, the series converges to the function plug in $x = \frac{1}{2}$, which is $\frac{3}{4}$.
So the correct answer is (D).

(4) 20 POINTS Explicitly show that the eigenvalue problem

$$\sqrt{1+x^2}\phi'' + x\phi' = -\lambda \cdot 3\sqrt{1+x^2}\phi \quad \text{on} \quad [0,1] \quad \text{with} \quad \phi(0) = \phi(1) = 0,$$

is a regular Sturm-Liouville problem. Write down the orthogonality condition on the eigenfunctions, and an asymptotic expression for the eigenvalues, valid as $\lambda \to \infty$.

Answer 4.
We want to multiply the equation by $f(x)$ so that it becomes a standard Sturm-Liouville equation, this requires $f(x)$ to satisfy

$$\left(\sqrt{1+x^2}f(x)\right)' = xf(x)$$

which is

$$\sqrt{1+x^2}f'(x) + \frac{x}{\sqrt{1+x^2}}f(x) = xf(x)$$

so

$$\frac{f'(x)}{f(x)} = \frac{x}{\sqrt{1+x^2}} - \frac{x}{1+x^2}$$

notice the left hand side is $(\ln f(x))'$. Integrating about x, we get

$$\ln f(x) = \int \Big(\frac{x}{\sqrt{1+x^2}} - \frac{x}{1+x^2}\Big)dx = \sqrt{1+x^2} - \frac{1}{2}\ln(1+x^2)$$

and

$$f(x) = e^{\sqrt{1+x^2}-\frac{1}{2}\ln(1+x^2)} = \frac{e^{\sqrt{1+x^2}}}{\sqrt{1+x^2}}$$

Multiply the original differential equation by $f(x)$, we get

$$e^{\sqrt{1+x^2}}\phi'' + \frac{xe^{\sqrt{1+x^2}}}{\sqrt{1+x^2}}\phi' + \lambda 3e^{\sqrt{1+x^2}}\phi = 0$$

therefore

$$p(x) = e^{\sqrt{1+x^2}}, \quad q(x) = 0, \quad \sigma(x) = 3e^{\sqrt{1+x^2}}$$

we see that p, q, σ are continuous in $[0,1]$, and $p(x) > 0$, $\sigma(x) > 0$ as they are exponential functions, also the boundary conditions are homogeneous, so this is a regular Sturm-Liouville equation.
If we denote ϕ_n as the n-th eigenfunction, then the orthogonality condition is

$$\int_0^1 \phi_n\phi_m 3e^{\sqrt{1+x^2}}dx = 0, \quad m \neq n$$

Denote λ_n as the n-th eigenvalue, then as $n \to \infty$

$$\lambda_n \sim \left(\frac{n\pi}{\int_0^1 \left(\frac{\sigma(x)}{p(x)}\right)^{\frac{1}{2}}dx}\right)^2 = \left(\frac{n\pi}{\int_0^1 \sqrt{3}dx}\right)^2 = \frac{n^2\pi^2}{3}$$

(5) 20 POINTS Let

$$\sum_{n=-\infty}^{\infty} c_n e^{-inx}$$

be the complex form of the Fourier series of the function

$$f(x) = x + 1$$

on the interval $[-\pi, \pi]$. What is the value of the sum $c_{-2} + c_1$?

Answer 5.
For $n \neq 0$, we have

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(x+1)e^{inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}(x+1)\frac{1}{in}de^{inx} \\
&= \frac{1}{2n\pi i}(x+1)e^{inx}\Big|_{-\pi}^{\pi} - \frac{1}{2n\pi i}\int_{-\pi}^{\pi}e^{inx}dx \\
&= \frac{1}{2n\pi i}(\pi+1)e^{in\pi} - \frac{1}{2n\pi i}(-\pi+1)e^{-in\pi} - \frac{1}{2n\pi i}\int_{-\pi}^{\pi}e^{inx}dx \\
&= \frac{2\pi e^{in\pi}}{2n\pi i} - \frac{1}{2n\pi i}\int_{-\pi}^{\pi}e^{inx}dx \\
&= \frac{e^{in\pi}}{ni} - \frac{1}{2n^2\pi i^2}e^{inx}\Big|_{-\pi}^{\pi} \\
&= \frac{e^{in\pi}}{ni}
\end{aligned}$$

So

$$c_1 = \frac{e^{i\pi}}{i} = \frac{-1}{i} = i$$

$$c_{-2} = \frac{e^{-2\pi i}}{-2i} = \frac{1}{-2i} = \frac{i}{2}$$

and

$$c_{-2} + c_1 = \frac{i}{2} + i = \frac{3i}{2}$$