# A Partial Table of Integrals

$$\int_0^x u\cos nu\;du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\;du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\;du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\;du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\;du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\;du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\;du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

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# Formulas Involving Bessel Functions

- Bessel's equation: $r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cJ_n(\alpha r)$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

$J_0(0) = 1$, $J_n(0) = 0$ if $n > 0$. $z_{nm}$ is the $m$th positive zero of $J_n(x)$.

- Orthogonality relations:

$$\text{If } m \neq k \text{ then } \int_0^1 xJ_n(z_{nm}x)J_n(z_{nk}x)\,dx = 0 \quad \text{and} \quad \int_0^1 x(J_n(z_{nm}x))^2\,dx = \frac{1}{2}J_{n+1}(z_{nm})^2.$$

- Recursion and differentiation formulas:

$$\frac{d}{dx}(x^nJ_n(x)) = x^nJ_{n-1}(x) \quad \text{or} \quad \int x^nJ_{n-1}(x)\,dx = x^nJ_n(x) + C \quad \text{for } n \geq 1 \tag{1}$$

$$\frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1}(x) \quad \text{for} \quad n \geq 0 \tag{2}$$

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x) \tag{3}$$

$$J_n'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x) \tag{4}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \tag{5}$$

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{6}$$

- Modified Bessel's equation: $r^2R'' + rR' - (\alpha^2r^2 + n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cI_n(\alpha r)$.

$$I_n(x) = i^{-n}J_n(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

## Formulas Involving Associated Legendre and Spherical Bessel Functions

- Associated Legendre Functions: $\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\left(\mu-\frac{m^2}{\sin\phi}\right)g=0$. Using the substitution $x=\cos\phi$, this equation becomes $\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right)+\left(\mu-\frac{m^2}{1-x^2}\right)g=0$. This equation has bounded solutions only when $\mu=n(n+1)$ and $0\le m\le n$. The solution $P_n^m(x)$ is called an associated Legendre function of the first kind.

- Associated Legendre Function Identities:

$$P_n^0(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ and } P_n^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \text{ when } 1\le m\le n$$

- Orthogonality of Associated Legendre Functions: If $n$ and $k$ are both greater than or equal to $m$,

$$\text{If } n\neq k \text{ then } \int_{-1}^{1}P_n^m(x)P_k^m(x)dx=0 \text{ and } \int_{-1}^{1}\left(P_n^m(x)\right)^2dx=\frac{2(n+m)!}{(2n+1)(n-m)!}.$$

- Spherical Bessel Functions: $(\rho^2 f')'+(\alpha^2\rho^2-n(n+1))f=0$. If we define the spherical Bessel function $j_n(\rho)=\rho^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\rho)$, then only solution of this ODE bounded at $\rho=0$ is $j_n(\alpha\rho)$.

- Spherical Bessel Function Identity:

$$j_n(x)=x^2\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right).$$

- Spherical Bessel Function Orthogonality: Let $z_{nm}$ be the $m$-th positive zero of $j_m$.

$$\text{If } m\neq k \text{ then } \int_0^1 x^2 j_n(z_{nm}x)j_n(z_{nk}x)dx=0 \quad\text{and}\quad \int_0^1 x^2(j_n(z_{nm}x))^2dx=\frac{1}{2}(j_{n+1}(z_{nm}))^2.$$

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## One-Dimensional Fourier Transform

$$\mathcal{F}[u](\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}[U](x)=\int_{-\infty}^{\infty}U(\omega)e^{-i\omega x}d\omega$$

Table of Fourier Transform Pairs

| Fourier Transform Pairs ($\alpha>0$) | | Fourier Transform Pairs ($\beta>0$) | |
|---|---|---|---|
| $u(x)=\mathcal{F}^{-1}[U]$ | $U(\omega)=\mathcal{F}[u]$ | $u(x)=\mathcal{F}^{-1}[U]$ | $U(\omega)=\mathcal{F}[u]$ |
| $e^{-\alpha x^2}$ | $\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}$ | $\sqrt{\frac{\pi}{\beta}}e^{-\frac{x^2}{4\beta}}$ | $e^{-\beta\omega^2}$ |
| $e^{-\alpha\|x\|}$ | $\frac{1}{2\pi}\frac{2\alpha}{x^2+\alpha^2}$ | $\frac{2\beta}{x^2+\beta^2}$ | $e^{-\beta\|\omega\|}$ |
| $u(x)=\begin{cases}0 & \|x\|>\alpha\\ 1 & \|x\|<\alpha\end{cases}$ | $\frac{1}{\pi}\frac{\sin\alpha\omega}{\omega}$ | $2\frac{\sin\beta x}{x}$ | $U(\omega)=\begin{cases}0 & \|\omega\|>\beta\\ 1 & \|\omega\|<\beta\end{cases}$ |
| $\delta(x-x_0)$ | $\frac{1}{2\pi}e^{i\omega x_0}$ | $e^{-i\omega_0 x}$ | $\delta(\omega-\omega_0)$ |
| $\frac{\partial u}{\partial t}$ | $\frac{\partial U}{\partial t}$ | $\frac{\partial^2 u}{\partial t^2}$ | $\frac{\partial^2 U}{\partial t^2}$ |
| $\frac{\partial u}{\partial x}$ | $-i\omega U$ | $\frac{\partial^2 u}{\partial x^2}$ | $(-i\omega)^2 U$ |
| $xu$ | $-i\frac{\partial U}{\partial\omega}$ | $x^2u$ | $(-i)^2\frac{\partial^2 U}{\partial\omega^2}$ |
| $u(x-x_0)$ | $e^{i\omega x_0}U$ | $\frac{1}{2\pi}\int_{-\infty}^{\infty}f(s)g(x-s)ds$ | $FG$ |