Practice problems for the First Midterm, Math 241, Fall 2013

Question 1. Let $u(x, y)$ be a function of two variables satisfying the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Suppose that u is a radial function, that is: in polar coordinates, the function u depends on the coordinate r but is indepenendant of the coordinate θ. Suppose also that $u(1, 0) = 0$ and $u(0, e) = 1/2$. What is the value of $u(e^2, 0)$?

(A) $e/2$	(B) $e + 1$	(C) 1
(D) e	(E) e^2	(F) $e^2 + 1$

Question 2. Heat is flowing through a thin wire of length 2 meters, so that one end of the wire is at $x = 0$ and the other end is at $x = 2$. Let $u(x, t)$ be the temeperature at point x and time t, and suppose $u(x, t)$ satisfies the inhomogeneous heat equation

$$u_t = \frac{1}{4} u_{xx} + 4 - x^2, \qquad \text{for } 0 < x < 2, \text{ and } t > 0.$$

If the boundary conditions are

$$u(0, t) = 0 \quad \text{and} \quad u_x(2, t) = 0,$$

then what is

$$\lim_{t \to +\infty} u(2, t),$$

that is what is the equilibrium temperature at the insulated end of the wire?

(A) 0	(B) 1/2	(C) 16
(D) 4	(E) 8/3	(F) 6

Question 3. Let $u(x,t)$ be the solution of the equation:

$$u_t = u_{xx} \quad \text{for } 0 < x < 4 \text{ and } t > 0$$

satisfying the boundary conditions

$$u(0,t) = 0 \text{ and } u(4,t) = 0$$
$$u(x,0) = 3\sin 8\pi x$$

What is $u\left(\frac{3}{16}, 3\right)$?

(A) $3e^{-\pi^2}$ (B) $3e^{-64\pi^2}$ (C) $3e^{-192\pi^2}$
(D) $-3e^{-\pi^2}$ (E) $-3e^{-64\pi^2}$ (F) $-3e^{-192\pi^2}$

Question 4. If, for $0 \leq x \leq 5$, we have

$$\sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi x}{5}\right) = 5x - x^2,$$

what is the value of b_3?

(A) $\dfrac{2}{9\pi^2}$ (B) $\dfrac{20}{9\pi^2}$ (C) $\dfrac{200}{27\pi^3}$

(D) $\dfrac{1000}{27\pi^3}$ (E) $\dfrac{100}{81\pi^4}$ (F) $\dfrac{2000}{81\pi^4}$

Question 5. Suppose the heat flux at every point of the outer circle of an annulus with inner radius $R_1 = 2$ and outer radius $R_2 = 5$ points directly out of the annulus and has magnitude 4. Also, suppose that at every point of the inner circle of the annulus the flux points directly into the annulus and has the same magnitude all around the circle. What must the magnitude of this latter flux be so that the temperature of the annulus will be at equilibrium? (In other words, so that there is a solution of the Laplace equation with these flux values.)

(A) -4	(B) 0	(C) 4
(D) -10	(E) 10	(F) 8

Question 6.

(a) Compute the Fourier cosine series for the function $f(x) = x^2$ on the interval $[0, \pi]$.

(b) Fully simplify your answer - the formula for the coefficients should not involve sines or cosines.

(c) Does the Fourier cosine series converge to the function f at the point $x = 0$? Justify your answer.

Question 7. Consider the following BVP posed for $0 < x < L$ and $t > 0$:

$$\text{PDE:} \quad u_t = u_{xx} + 2u_x + u$$

$$\text{BC:} \quad u(0,t) = 0, \quad \text{and} \quad u(L,t) + u_x(L,t) = 0$$

Apply the method of separation of variables to determine what ordinary differential equations are implied for functions of x and t and what boundary conditions (if any) are necessary for each of those ODEs. You do **not** need to solve these ODEs.

Question 8. Show that for $0 < x < 1$ and $t > 0$, the function

$$u(x,t) = e^{-\frac{x^2}{4t+4}}(t+1)^{-\frac{1}{2}}$$

is a solution of the heat equation

$$u_t = u_{xx}.$$

Give one example of a homogeneous boundary condition that this solution satisfies on the specified domain.